From the Quintic Equation to the Yang Baxter Equation

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Polynomial equations are used in almost every branch of mathematics. By the nineteenth century, mathematicians had already discovered ways to solve quadratic, cubic and quartic equations and turned their attention to solving quintic equations. In 1796 Ruffini proposed that quintic equations could not be solved by radicals and in 1799 gave an incomplete proof. In 1824 Abel presented a proof. Abel-Ruffini theroem states that there is no closed formula solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients. Galois (1811-1832) took up Abel's work and came up with the key idea that every polynomial has a symmetry group which determines whether it can be solved by radicals.

Galois groups provide a powerful perspective from which to study polynomial equations, dealing with abstract algebraic structures instead of manipulating explicit algebraic expressions, emphasizes abstraction over calculation. Developing Galois theory using abstract algebraic structures helps us to see its connections to other parts of mathematics, so that we can study various mathematical objects in algebra and number theory in ways that are not easy to describe in the original language of polynomials. In many relevant theories developed during the twentieth century, Galois representations play the essential role of connecting geometric and analytic objects.

A group action is a representation of the elements of a group as symmetries of a set. In this sense, Galois group becomes the group of automorphisms of the splitting field of the polynomial. By scalar extension, we have the elements of a group algebra $k[G]$ acting as endomorphisms of a field extension K/k. We make abstraction of polynomials with the definition of Galois field extensions.

Hopf Galois theory, introduced by Chase and Sweedler in 1969, expands the classical Galois theory by considering the Galois property in terms of the action of the group algebra $k[G]$ on K/k and then replacing it by the action of a Hopf algebra. A Hopf algebra is an algebra together with all necessary extra ingredients to define the tensor product of two representations as a representation (comultiplication) and the dual of a representation as a representation (the antipode map). Under this generalization of Galois theory, many mathematical results can be revisited and reformulated. Up to now, this has been done mostly in the context of Galoi module theory and the existence of normal basis.

In the case of separable extensions, Greither and Pareigis proved in 1987 that the Hopf Galois property admits a group-theoretical formulation suitable for counting and classifying, and also to perform explicit descriptions of computations. In this scenario, we play with two finite groups N and G such that N is a regular subgroup of $\mathrm{Perm}(G)$. This relationship may be reversed under the so-called Byott's reformulation to get G regular subgroup of the holomorph $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}N$.

On the other hand, the Yang-Baxter equation first appeared in theoretical physics in 1967, in a paper by Yang, and in statistical mechanics around 1980, in Baxter's work. Later, it turned out that this equation plays a crucial role in quantum groups, knot theory, braided categories, analysis of integrable systems, quantum mechanics, non-commutative descent theory, quantum computing, non-commutative geometry, etc. For a first glimpse about the connection of both worlds we can think of quantum group as a particular case of quasitriangular (or braided) Hopf algebra. From the axioms of this structure it follows that its universal R-matrix satisfies the Yang-Baxter equation.

In 1990 Drinfeld suggested the idea of set-theoretical solutions of the Yang-Baxter equation.From such a solution we can construct a permutation group called involutive Yang-Baxter group obtaining a bijective correspondence. In 2007 Rump introduced a new algebraic structure called (left)brace, a generalized radical ring where set theoretical solutions can be embedded to study its interaction. In Rump's words, he establishes a Galois theory between ideals of braces and quotient set theoretical solutions. Braces allow us to use ring-theoretic and group-theoretic methods to study solutions to the Yang-Baxter equation.

In 2016, Bachiller proves that a finite group is an involutive Yang-Baxter group if and only if it is the multiplicative group of a left brace. Bachiller proves that given an abelian group N, there is a bijective correspondence between left braces with additive group N and regular subgroups of $\mathrm{Hol}(N)$ such that isomorphic left braces correspond to conjugate subgroups of $\mathrm{Hol}(N)$ by elements of $\mathrm{Aut}(N)$. In this way he established the connection between braces and Hopf-Galois separable extensions.

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