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From Witt rings of rings to Witt groups of exact categories with duality

A **sheaf of algebras** $\mathcal{O}_X$ on a topological space X is an assignment of an algebra $\mathcal{O}_X(U)$ called a **local section** to each open set U in X, together with, for each inclusion $U \subseteq V$, a **restriction homomorphism** $\mathrm{res}^{\mathcal{O}_X}_{V,U} \colon \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ such that:

i. $\mathrm{res}^{\mathcal{O}_X}_{U,U} = \mathrm{id}_U$;

ii. if $U \subseteq V \subseteq W$, then $\mathrm{res}^{\mathcal{O}_X}_{V,U} \circ \mathrm{res}^{\mathcal{O}_X}_{W,V} = \mathrm{res}^{\mathcal{O}_X}_{W,U}$;

iii. for each open cover $\{U_i |\, i \in I\}$ of $U \subseteq X$ and for each collection of elements $f_i \in \mathcal{O}_X(U_i)$, $i \in I$, if:

$$\mathrm{res}^{\mathcal{O}_X}_{U_i, U_i \cap U_j}(f_i) = \mathrm{res}^{\mathcal{O}_X}_{U_j, U_i \cap U_j}(f_j),$$

for all $i, j \in I$, then there is a unique $f \in \mathcal{O}_X(U)$ such that:

$$f_i = \mathrm{res}^{\mathcal{O}_X}_{U, U_i}(f),$$

for all $i \in I$.

A structure which only satisfies conditions i. and ii. will be called a **presheaf**.

In particular, we define in this way sheafs of groups, sheafs of rings, sheafs of modules etc.

A topological space equipped with a sheaf of rings is called a **ringed space**.

Example 1. Let M be a smooth manifold i.e. a topological space M equipped with a maximal differentiable atlas. Then, for each open set U of M, the set $C(U)$ of real-valued continuous functions on U with point-wise addition and multiplication is a ring. If $V \subseteq U$ then the restriction homomorphism $C(U) \to C(V)$ is given by actually restricting functions. It is easy to verify that this is indeed a sheaf: it is one of the prototypical examples that shall serve as a basis for much of the intuition.

We think of elements of $\mathcal{O}_X(U)$ as "functions" defined on U. With this intuition axioms of a sheaf can be understood in the following sense:

- restricting a function to its original domain does nothing;
- restricting and then restricting again is the same as restricting all at once;
- if we have two functions defined on some different open sets, and these functions agree on the overlaps, then we can glue them together to get a unique new function on the union of these open sets, and if we restrict this glueing back to one of the open sets, we get the corresponding function back.

Let X be a topological space and $\mathcal{O}_X$ a sheaf of algebras on X. Let $x \in X$ be a point. A **stalk** of $\mathcal{O}_X$ at x is the set

$$\mathcal{O}_{X,x} = \{(U, f) | x \in X, f \in \mathcal{O}_X(U)\} / \sim$$

where the relation $\sim$ is defined as follows:

$$(U, f) \sim (V, g) \quad \Leftrightarrow \quad \text{there is an open set } W \subseteq U \cap V \text{ such that } f\restriction_W = g\restriction_W$$

By abuse of the notation we shall often denote (U, f) in $\mathcal{O}_{X,x}$ by f_x or even f. Also, we will say $f = g$ in $\mathcal{O}_{X,x}$ for two local sections of $\mathcal{O}_X$ defined in an open neighborhood of x to denote that they have the same image in $\mathcal{O}_{X,x}$.

Example 2. Let $X = \mathbb{R}^n$ with the Euclidean topology and define the sheaf of $\mathcal{C}^\infty$ functions by taking as $\mathcal{C}^\infty(U)$ the set of all $\mathcal{C}^\infty$-functions $f\colon U \to \mathbb{R}$. Let $x \in \mathbb{R}^n$. An element of a stalk $\mathcal{C}^\infty_{X,x}$ is defined by a function f whose domain contains x. Two functions determine the same element in this stalk if they agree on a neighborhood of x. The stalk $\mathcal{C}^\infty_{X,x}$ is what analysts call the **germ** of a $\mathcal{C}^\infty$-function at x.

The canonical map $\mathcal{O}_X(U) \to \prod_{x \in U} \mathcal{O}_{X,x}$, for each open subset U of X given by

$$f \mapsto \prod_{x \in U} (U, f)$$

is injective.

- A.J. de Jong, *Stacks Project*, Chapter 6 “Sheaves on spaces”, Section 11 “Stalks”, `https://stacks.math.columbia.edu/tag/0078`

Let X be a topological space, and $\mathcal{O}_X, \mathcal{O}'_X$ be two sheaves of algebras on X. Then a **morphism** $\varphi\colon \mathcal{O}_X \to \mathcal{O}'_X$ is a collection of homomorphisms of algebras $\varphi\colon \mathcal{O}_X(U) \to \mathcal{O}'_X(U)$, one for each open set $U \subseteq X$, which commute with the restriction maps i.e. if $V \subseteq U \subseteq X$ are open sets, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\varphi(U)} & \mathcal{O}'_X(U) \\ \downarrow{\scriptstyle \mathrm{res}^{\mathcal{O}_X}_{U,V}} & & \downarrow{\scriptstyle \mathrm{res}^{\mathcal{O}'_X}_{U,V}} \\ \mathcal{O}_X(V) & \xrightarrow[\varphi(V)]{} & \mathcal{O}'_X(V) \end{array}$$

Proposition 3. *A sheaf on a topological space X is completely determined by its values on basic open sets.*

- J.-P. Serre, *FAC (Faisceaux Algebriques Coherents)*, Proposition I.1.4.4. English translation (with a lot of background material) by Andy McLennan, `https://andymclennan.droppages.com/fac_trans.pdf`
- A.J. de Jong, *Stacks Project*, Chapter 6 "Sheaves on spaces", Section 30 "Bases and sheaves", `https://stacks.math.columbia.edu/tag/009H`

Example 4. Let R be a commutative ring with 1. We define the **Zariski topology** on $\mathrm{Spec}(R)$ by declaring basic open sets to be:

$$\mathrm{Spec}(R)_f = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\},$$

for $f \in R$. The **structure sheaf** of $\mathrm{Spec}(R)$ is the sheaf defined by

$$\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)_f) = R_f$$

and for $\mathrm{Spec}(R)_f \subseteq \mathrm{Spec}(R)_g$ the restriction $\mathrm{res}^{\mathcal{O}_{\mathrm{Spec}(R)}}_{\mathrm{Spec}(R)_g, \mathrm{Spec}(R)_f} \colon R_g \to R_f$ is the induced homomorphism of R-algebras.

When we think of elements of R as functions on $\mathrm{Spec}(R)$ we mean $f(\mathfrak{p})$ to be the image of $f \in R$ under the canonical map

$$R \to R/\mathfrak{p} \to \mathrm{ff}(R/\mathfrak{p})$$

This, however, is not very rigorous, as we do not always yield an actual function: take $R = \mathbb{Z}$ and $f = 7$ – then $f(\langle 2 \rangle) = 1$ in $\mathbb{Z}_2$ and $f(\langle 5 \rangle) = 2$ in $\mathbb{Z}_5$, so these values lie in different fields.

The ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called an **affine scheme**.

A **scheme** is a ringed space $(X, \mathcal{O}_X)$ which is locally affine in the following sense: there exists an open covering $\{U_i | \, i \in I\}$ of X such that the restriction of $\mathcal{O}_X$ to each U_i is an affine scheme.

For a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-**module** (or a **sheaf of** $\mathcal{O}_X$-**modules**) is a sheaf of modules $\mathcal{M}$ over the topological space X such that for each open set U of X, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and the restrictions in $\mathcal{M}$ are compatible with the restrictions in $\mathcal{O}_X$ in the following sense: for each inclusion $U \subseteq V$:

$$\mathrm{res}^{\mathcal{M}}_{V,U}(a \cdot m) = \mathrm{res}^{\mathcal{O}_X}_{V,U}(a) \cdot \mathrm{res}^{\mathcal{M}}_{V,U}(m),$$

for $a \in \mathcal{O}_X(U)$, $m \in \mathcal{M}(U)$.

For a scheme $(X, \mathcal{O}_X)$ a **vector bundle** is an $\mathcal{O}_X$-module $\mathcal{E}$ which is locally free of finite rank, that is, for every point $x \in X$, there exists an open neighbourhood U of x such that $\mathcal{E}(U)$ is an $\mathcal{O}_X(U)$-module isomorphic to the $\mathcal{O}_X(U)$-module $(\mathcal{O}_X(U))^{n_x}$, for some $n_x \in \mathbb{N}$. We call n_x the **local rank** of $\mathcal{E}$ at x, and the locally constant function $x \mapsto n_x$ on X the **rank** $\operatorname{rk} \mathcal{E}$ of $\mathcal{E}$.

Remark 5. This is not the most general definition, but one tailored to our needs. Proper definitions are to be found in:

- J. Dieudonne, A. Grothendieck, *EGA I (Elements de geometrie algebrique I)*, Definition 1.7.8
- A.J. de Jong, *Stacks Project*, Chapter 27 "Constructions of Schemes", Section 6 "Vector bundles", `https://stacks.math.columbia.edu/tag/01M1`

Example 6. Let R be a commutative ring with 1 and $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ the affine scheme.

Every vector bundle $\mathcal{E}$ on $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is canonically isomorphic to the R-module $\mathcal{E}(X)$ (*EGA I*, Section 1.4) i.e. the category of vector bundles over $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is equivalent to the category of finitely generated projective modules over R.

For a scheme $(X, \mathcal{O}_X)$ a **symmetric bilinear form** β on a vector bundle $\mathcal{E}$ is a morphism β: $\mathcal{E} \times \mathcal{E} \to \mathcal{O}_X$ in the category of sheaves over X such that for every open set U in X the map

$$\beta(U): \mathcal{E}(U) \times \mathcal{E}(U) \to \mathcal{O}_X(U)$$

is a symmetric bilinear form of the $\mathcal{O}_X(U)$-module $\mathcal{E}(U)$.

For a scheme $(X, \mathcal{O}_X)$ and a vector bundle $\mathcal{E}$ we define the **dual vector bundle** $\mathcal{E}^*$ as follows: for every open set U in X, $\mathcal{E}^*(U)$ is the set of $\mathcal{O}_X(U)$-module homomorphisms $\mathcal{E}(U) \to \mathcal{O}_X(U)$. One checks that this is, indeed, a vector bundle.

For a scheme $(X, \mathcal{O}_X)$ and a symmetric bilinear form β on a vector bundle $\mathcal{E}$, for each open set U in X the form β defines a map $\mathcal{E}(U) \to \mathcal{E}^*(U)$ in a natural way by assigning $\mathcal{E}(U) \ni u \mapsto \beta(u, \cdot) \in \mathcal{E}^*(U)$. These maps together constitute a homomorphism φ from a bundle $\mathcal{E}$ to $\mathcal{E}^*$. We call β **non-degenerate** if φ is an isomorphism.

A pair $(\mathcal{E}, \beta)$ consisting of a vector bundle $\mathcal{E}$ and a symmetric bilinear form β will be called a **bilinear bundle**.

If β is non-degenerate, we will call $(\mathcal{E}, \beta)$ a **bilinear space**.

A **morphism** between bilinear bundles $\varphi\colon (\mathcal{E}_1, \beta_1) \to (\mathcal{E}_2, \beta_2)$ is a morphism of vector bundles $\varphi\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that, for each open set U in X:

$$\beta_2(U)(\varphi(U)(u), \varphi(U)(v)) = \beta_1(U)(u, v),$$

for each $u, v \in \mathcal{E}_1(U)$.

For a scheme $(X, \mathcal{O}_X)$ and bilinear bundles $(\mathcal{E}_1, \beta_1)$, $(\mathcal{E}_2, \beta_2)$ we define the **orthogonal sum** $(\mathcal{E}_1 \oplus \mathcal{E}_2, \beta_1 \oplus \beta_2)$ as follows: the $\mathcal{O}_X$-module $\mathcal{E}_1 \oplus \mathcal{E}_2$ is the sheaf such that for each open set U in X:

$$\mathcal{E}_1 \oplus \mathcal{E}_2(U) = \mathcal{E}_1(U) \oplus \mathcal{E}_2(U),$$

and $\beta_1 \oplus \beta_2 : \mathcal{E}_1 \oplus \mathcal{E}_2 \times \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{O}_X$ is the morphism such that for each open set U in X:

$$\beta_1 \oplus \beta_2(U)(u_1 \oplus u_2, v_1 \oplus v_2) = \beta_1(U)(u_1, v_1) + \beta_2(U)(u_2, v_2),$$

for $u_1, v_1 \in \mathcal{E}_1(U)$ and $u_2, v_2 \in \mathcal{E}_2(U)$.

Assigning for an open subset U in X the module $\mathcal{E}_1(U) \otimes_{\mathcal{O}_{X(U)}} \mathcal{E}_2(U)$ does not neccesarily yield a sheaf – we only get a presheaf, which needs to undergo a **sheafification**. We shall briefly describe this process.

Let X be a topological space and $\mathcal{O}_X$ a presheaf. For every open subset U of X define

$$\mathcal{O}_X^{\#}(U) = \left\{ (f_x) \in \prod_{x \in U} \mathcal{O}_{X,x} \,|\, (f_x) \text{ satisfies the condition } (*) \right\}$$

where the condition (*) is the following one:

> for every $x \in U$ there exists an open neighborhood $x \in V \subset U$ and $g \in \mathcal{O}_X(V)$ such that for all $y \in V$ we have $f_y = (V, g)$ in $\mathcal{O}_{X,y}$

The presheaf $\mathcal{O}_X^{\#}$ is, in fact, a sheaf called the sheafification of $\mathcal{O}_X$.

- A.J. de Jong, *Stacks Project*, Chapter 6 “Sheaves on spaces”, Section 17 “Sheafification”, `https://stacks.math.columbia.edu/tag/007X`

In particular, the **tensor product** of two bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ is the sheafification of the presheaf $\mathcal{E}_1(U) \otimes_{\mathcal{O}_{X(U)}} \mathcal{E}_2(U)$ denoted by $\mathcal{E}_1 \otimes \mathcal{E}_2$.

If U is an affine open subset in X, then

$$\mathcal{E}_1 \otimes \mathcal{E}_2(U) = \mathcal{E}_1(U) \otimes_{\mathcal{O}_{X(U)}} \mathcal{E}_2(U)$$

- Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press 2002, Proposition 5.1.12 (b)

For a scheme $(X, \mathcal{O}_X)$ and bilinear bundles $(\mathcal{E}_1, \beta_1)$, $(\mathcal{E}_2, \beta_2)$ we define the **tensor product** $(\mathcal{E}_1 \otimes \mathcal{E}_2, \beta_1 \otimes \beta_2)$ by defining $\beta_1 \otimes \beta_2 \colon \mathcal{E}_1 \otimes \mathcal{E}_2 \times \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{O}_X$ as the unique morphism such that for each open set U in X:

$$\beta_1 \otimes \beta_2(U)(u_1 \otimes u_2, v_1 \otimes v_2) = \beta_1(U)(u_1, v_1) \cdot \beta_2(U)(u_2, v_2),$$

for $u_1, v_1 \in \mathcal{E}_1(U)$ and $u_2, v_2 \in \mathcal{E}_2(U)$.

For any modules M, N we have $(M \oplus N)^* \cong M^* \oplus N^*$, so that for bilinear bundles $(\mathcal{E}_1, \beta_1)$,$(\mathcal{E}_2, \beta_2)$ we have $(\mathcal{E}_1 \oplus \mathcal{E}_2)^* \cong \mathcal{E}_1^* \oplus \mathcal{E}_2^*$, and if φ_1, φ_2 denote the homomorphisms $\mathcal{E}_1 \ni u \mapsto \beta_1(u, \cdot) \in \mathcal{E}_1^*$, $\mathcal{E}_2 \ni v \mapsto \beta_2(v, \cdot) \in \mathcal{E}_2^*$ defined by β_1, β_2, respectively, then $\beta_1 \oplus \beta_2$ corresponds to the homomorphism

$$\begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} : \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}_1^* \oplus \mathcal{E}_2^*.$$

For finitely generated projective modules M, N we also have $(M \otimes N)^* \cong M^* \otimes N^*$, so that for bilinear bundles $(\mathcal{E}_1, \beta_1)$,$(\mathcal{E}_2, \beta_2)$ we have $(\mathcal{E}_1 \otimes \mathcal{E}_2)^* \cong \mathcal{E}_1^* \otimes \mathcal{E}_2^*$, and if φ_1, φ_2 denote the homomorphisms $\mathcal{E}_1 \ni u \mapsto \beta_1(u, \cdot) \in \mathcal{E}_1^*$, $\mathcal{E}_2 \ni v \mapsto \beta_2(v, \cdot) \in \mathcal{E}_2^*$ defined by β_1, β_2, respectively, then $\beta_1 \otimes \beta_2$ corresponds to the homomorphism

$$\varphi_1 \otimes \varphi_2 : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}_1^* \oplus \mathcal{E}_2^*.$$

- K. Szymiczek, *Witt ring of a commutative ring*

A **category with duality** is a triple $(\mathcal{C}, ^*, \bar{\omega})$ made of category $\mathcal{C}$, an involutive endo-functor $^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with given isomorphism $\bar{\omega}\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cong} {}^*\circ{}^*$ such that $M \mapsto M^*$ is a functor and $\bar{\omega}_M\colon M \xrightarrow{\cong} (M^*)^*$ is a natural isomorphism such that

$$(\bar{\omega}_M)^* \circ \bar{\omega}_{M^*} = \mathrm{id}_{M^*}$$

for any object M of $\mathcal{C}$.

A **symmetric space** in $(\mathcal{C}, ^*, \bar{\omega})$ consists of a pair (P, φ), where P is an object in $\mathcal{C}$, and φ: $P \xrightarrow{\cong} P^*$ is an isomorphism called **symmetric form** which is symmetric in the following sense:

$$\varphi^* \circ \bar{\omega}_P = \varphi$$

i.e. $\varphi^* = \varphi$ when P is identified with P^{**} via $\bar{\omega}_P$.

Example 7. Let X be a scheme, $\mathcal{C}$ the category of vector bundles in X, ${}^*: \mathcal{C} \to \mathcal{C}$ the duality defined by

$$E^* = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$$

and $\bar{\omega}: E \xrightarrow{\cong} E^{**}$ the natural identification defined in the usual way.

Example 8. Let X be a scheme, $\mathcal{L}$ a fixed line bundle in X, i.e. a vector bundle of rank 1, $\mathcal{C}$ the category of vector bundles in X, ${}^*\colon \mathcal{C} \to \mathcal{C}$ the usual duality twisted by the line bundle $\mathcal{L}$, i.e.

$$E^* = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{L}) \cong \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}$$

and $\bar{\omega}$ defined in the usual way. For $\mathcal{L} = \mathcal{O}_X$ we get the usual dual.

- M. Zibrovius, *Witt groups of curves and surfaces*

Two symmetric spaces (P, φ) and (Q, ψ) are **isometric** if there exists an isomorphism $h\colon P \xrightarrow{\cong} Q$ in the category $\mathcal{C}$ such that

$$h^*\psi h = \varphi$$

(note that $h^*\colon Q^* \to P^*$ is given by $h^*(\phi) = \phi \circ h$).

A **morphism** of categories with duality $(\mathcal{C}, *^{\mathcal{C}}, \bar{\omega}^{\mathcal{C}}) \to (\mathcal{D}, *^{\mathcal{D}}, \bar{\omega}^{\mathcal{D}})$ consists of a pair (F, η) where $F: \mathcal{C} \to \mathcal{D}$ is a functor and $\eta: F \circ *^{\mathcal{C}} \xrightarrow{\cong} *^{\mathcal{D}} \circ F$ is an isomorphism respecting $\bar{\omega}$ i.e. making the following diagram commutative:

$$\begin{array}{ccc} F(M) & \xrightarrow{F(\bar{\omega}^{\mathcal{C}}_M)} & F(M^{**}) \\ {\scriptstyle \bar{\omega}^{\mathcal{D}}_{F(M)}} \downarrow & & \downarrow {\scriptstyle \eta_{M^*}} \\ F(M^{**}) & \xrightarrow[\eta^*_M]{} & F(M^*)^* \end{array}$$

An **additive category** is a category where:

- hom-sets are Abelian groups,
- composition of morphisms is bilinear,
- finite biproducts (i.e. simultaneously products and coproducts, denoted $\bigoplus$) exist.

Example 9. Abelian groups (addition of morphisms is given point-wise, biproducts are direct sums), modules over rings, vector bundles over a scheme.

An **additive category with duality** is a category with duality, where $*$ is an additive functor, i.e. $(A \oplus B)^* = A^* \oplus B^*$ via the natural isomorphism.

A **morphism** of additive categories with duality is a morphism F of categories with duality such that F is also additive.

For a scheme $(X, \mathcal{O}_X)$ and a vector bundle $\mathcal{E}$ assume that for exery open set Z in X there is an $\mathcal{O}_X(Z)$-submodule $\mathcal{V}(Z)$ of $\mathcal{E}(Z)$, and that for open subsets $Z' \subseteq Z$ the restriction map $\mathrm{res}^{\mathcal{E}}_{Z,Z'}\colon \mathcal{E}(Z) \to \mathcal{E}(Z')$ maps $\mathcal{V}(Z)$ to $\mathcal{V}(Z')$. If the functor $\mathcal{V}\colon Z \to \mathcal{V}(Z)$ on the category of open subsets of X fulfills the sheaf condition, we call $\mathcal{V}$ an $\boldsymbol{\mathcal{O}_X}$**-submodule** of $\mathcal{E}$.

An $\mathcal{O}_X$-submodule is called a **subbundle** if $\mathcal{V}$ is locally a direct summand of $\mathcal{E}$, that is:

$$\mathcal{E}(Z) \cong \mathcal{V}(Z) \oplus W$$

for every open subset Z of X.

Note that for a subbundle $\mathcal{V}$ the quotient $\mathcal{E}\,/\,\mathcal{V}$ (defined in an obvoius way) is again a vector bundle (direct summands of locally free $\mathcal{O}_Z$-modules of finite rank are locally free of finite rank).

For a subbundle $\mathcal{V}$ of a bilinear bundle $(\mathcal{E}, \beta)$ we define $\mathcal{V}^\perp$ as follows: for an open set Z of X let:

$$\mathcal{V}^\perp(Z) = \{s \in \mathcal{E}(Z) \mid \forall_{t \in \mathcal{V}(Z')} \beta(Z')(s,t) = 0 \text{ for every open subset } Z' \subseteq Z\}.$$

Proposition 10. $\mathcal{V}^\perp(Z) = \ker\left(\mathcal{E}(Z) \xrightarrow{\varphi} \mathcal{E}(Z)^* \xrightarrow{\iota^*} \mathcal{V}(Z)^*\right)$, *where* $\iota\colon \mathcal{V}(Z) \to \mathcal{E}(Z)$ *is the inclusion map, i.e.* ι^* *is the restriction of linear maps* $\mathcal{E}(Z) \to \mathcal{O}_X(Z)$ *to* $\mathcal{V}(Z) \to \mathcal{O}_X(Z)$.

A **totally isotropic subbundle** or a **sublagrangian** of $\mathcal{E}$ is a subbundle such that $\mathcal{V} \subseteq \mathcal{V}^{\perp}$.

For a bilinear bundle $(\mathcal{E}, \beta)$ consider the vector bundle $\mathcal{E} \oplus \mathcal{E}^*$. For every open subset Z of X, for $s, t \in \mathcal{E}(Z)$, $s^*, t^* \in \mathcal{E}(Z)^*$ let

$$B(s + s^*, t + t^*) = \beta(s, t) + t^*(s) + s^*(t)$$

This is a bilinear form, and its associated homomorphism $\varphi\colon \mathcal{E} \oplus \mathcal{E}^* \to (\mathcal{E} \oplus \mathcal{E}^*)^* = \mathcal{E}^* \oplus \mathcal{E}$ has matrix

$$\begin{bmatrix} \varphi & \mathrm{id} \\ \mathrm{id} & 0 \end{bmatrix}$$

We denote the space $(\mathcal{E} \oplus \mathcal{E}^*, B)$ by $M(\mathcal{E}, \beta)$ and call **split metabolic**.

In particular, when $\beta = 0$, we call $M(\mathcal{E}, 0)$ **hyperbolic** and denote $H(\mathcal{E})$.

A subbundle $\mathcal{V}$ is called a **lagrangian** if $\mathcal{V} = \mathcal{V}^{\perp}$

A space which has a lagrangian is called **metabolic**.

Clearly:

$$\text{hyperbolic} \Rightarrow \text{split metabolic} \Rightarrow \text{metabolic}$$

For fields split metabolic $\Rightarrow$ hyperbolic.

This is no longer true for rings (take $\left(\mathbb{Z}^2, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right)$ and see Prof. Szymiczek's notes for details).

One shows that for affine schemes metabolic $\Rightarrow$ split metabolic (see Knebusch)

This is no longer true for non-affine schemes (see Knus-Ojanguren for an example)

An **exact category** is an additive category $\mathcal{C}$ that contains a distinguished class $\mathcal{E}$ of triples of objects and arrows

$$M' \to M \to M''$$

such that

i. $\mathcal{E}$ is closed under isomorphisms and contains all triples of the form

$$M' \to M' \oplus M'' \to M''$$

ii. if $M \to M''$ is a secons arrow in a triple (i.e. it is an **admissable epimorphism**) and $N \to M''$ is any arrow, then their pullback is again an admissable epimorphism

iii. if $M' \to M$ is a first arrow in a triple (i.e. it is an **admissable monomorphism**) and $M' \to N$ is any arrow, then their pushout is again an admissable monomorphism

iv. admissable monomorphisms are kernels of their corresponding admissable epimorphisms

v. admissable epimorphisms are cokernels of their corresponding admissable monomorphisms

vi. composition of admissable monomorphisms (epimorphisms) is an admissable monomorphism (epimorphism)

- D. Quillen, *Higher algebraic K-theory*, Springer, 1972
- B. Keller, *Chain complexes and stable categories*, Manuscripta Math. 67 (1990), 379-417

Basic idea: encapsulate the concept of short exact sequences in abelian categories without the morphisms actually having kernels and cokernels.

An **exact functor** is one that sends admissable triples to admissable triples.

An **exact category with duality** is an additive category with duality such that the functor $*$ is exact.

Let $(\mathcal{E}, *, \bar{\omega})$ be an exact category with duality, let (P, φ) be a symmetric space in $\mathcal{E}$, let $\alpha\colon L \rightarrowtail P$ be an admissable monomorphism. Define

$$(L, \varphi)^{\perp} = \ker\left(P \xrightarrow{\varphi} P^{*} \xrightarrow{\alpha^{*}} L^{*}\right)$$

An **admissable sublagrangian** of a symmetric space (P, φ) is an admissable monomorphism $\alpha: L \to P$ such that φ vanishes on L and the induced monomorphism $\beta: L \to L^{\perp}$ is admissable.

An **admissable lagrangian** is when $L = L^{\perp}$ and β is an isomorphism.

A symmetric space is **metabolic** if it has an admissable lagrangian.

For an exact category with duality $(\mathcal{E}, *, \bar{\omega})$ denote by $\mathrm{MW}(\mathcal{E}, *, \bar{\omega})$ the set of isometry classes of symmetric spaces, and by $\mathrm{NW}(\mathcal{E}, *, \bar{\omega})$ the subset of classes of metabolic spaces. The Witt group of $(\mathcal{E}, *, \bar{\omega})$ is the quotient

$$W(\mathcal{E}, *, \bar{\omega}) = \frac{\mathrm{MW}(\mathcal{E}, *, \bar{\omega})}{\mathrm{NW}(\mathcal{E}, *, \bar{\omega})}$$

in the sense explained by the following Remark.

Remark 11. Let $(M, +)$ be an Abelian monoid, and $N \subseteq M$ a submonoid. For $m_1, m_2 \in M$ define

$$m_1 \sim m_2 \Leftrightarrow \exists n_1, n_2 \in N\, [m_1 + n_1 = m_2 + n_2]$$

Then $\sim$ is an equivalence, and the set of equivalence classes $M\,/\,N$ inherits a structure of Abelian monoid via

$$[m_1] + [m_2] = [m_1 + m_2]$$

If for any element $m \in M$ there is an element $m' \in M$ such that $m + m' \in N$, then $M\,/\,N$ is an Abelian group with $-[m] = [m']$. It is canonically isomorphic to the quotient of the Grothendieck group of M by the subgroup generated by N.