List 4: Applications of Greither-Pareigis theorem

1. Let L/K be a (G, G')-separable Hopf-Galois extension and let N_1, N_2 be regular and G-stable subgroups of $\mathrm{Perm}(G/G')$. Consider the action $*$ of G on N_1 and N_2 defined as conjugation by $\lambda(G)$. Show that $N_1 \cong N_2$ as G-groups (that is, there is a G-equivariant group isomorphism between them) if and only if $\widetilde{L}[N_1]^G \cong \widetilde{L}[N_2]^G$ as K-Hopf algebras.

2. Two Hopf-Galois structures $(H, \cdot)$, $(H', \cdot')$ on the same field extension L/K are said to be isomorphic if there is an isomorphism of K-Hopf algebras $f\colon H \longrightarrow H'$ such that $h \cdot \alpha = f(h) \cdot' \alpha$. In practice, isomorphic Hopf-Galois structures on L/K are considered as the same Hopf-Galois structure (for instance, in the Greither-Pareigis theorem).

   (a) Let L/K be a Galois extension with group G. Prove that the classical Galois structure on L/K and the Hopf-Galois structure corresponding to $\rho(G)$ under the Greither-Pareigis correspondence are isomorphic.

   (b) Give an example of separable Hopf-Galois extension L/K and different (non-isomorphic) Hopf-Galois structures H and H' on L/K such that $N \cong N'$, where N (resp. N') is the permutation subgroup corresponding to H (resp. H').

3. Let L/K be a Galois extension with group G and let N be a regular and G-stable subgroup of $\mathrm{Perm}(G)$. Show that for each $\varphi \in \mathrm{Aut}(G)$, $(\varphi \circ N \circ \varphi^{-1})^{\mathrm{opp}} = \varphi \circ N^{\mathrm{opp}} \circ \varphi^{-1}$.

4. Let L/K be a Galois extension with group G. For each regular and G-stable subgroup N of $\mathrm{Perm}(G)$ and each $g \in G$, call $N_g := \rho(g)N\rho(g^{-1})$. Two Hopf-Galois structures on L/K corresponding to permutation subgroups N, N' are said to be ρ-conjugate if $N' = N_g$ for some $g \in G$. Fix such a subgroup N of $\mathrm{Perm}(G)$ and $g \in G$.

   (a) Prove that N_g is indeed a regular and G-stable subgroup of $\mathrm{Perm}(G)$.

   (b) Show that the map $\phi\colon L[N]^G \longrightarrow L[N_g]^G$ defined by $\phi(\sum_{\eta \in N} c_\eta \eta) = \sum_{\eta \in N} c_\eta \rho(g)\eta\rho(g^{-1})$ is an isomorphism of K-Hopf algebras.

   (c) Prove that $N_g^{\mathrm{opp}} = (N^{\mathrm{opp}})_g$.

5. Prove that every separable field extension of degree at most 4 is almost classically Galois.

6. Let L/K be a (G, G')-separable almost classically Galois extension and let J be a normal complement for G' in G. Write $*$ for the action of G on $M[J]$ defined as the Galois action on M and the conjugation by G on J. Show that if J is abelian, then there is an isomorphism of Hopf-Galois structures between

$$M[J]^{G'} = \{h \in M[J] \mid \tau * h = h \text{ for all } \tau \in G'\}$$

   together with its classical action on L and the Hopf-Galois structure on L/K corresponding to $\lambda(J)$.

7. Consider the extension L/K at Example 2.5.7. Determine explicitly the Hopf-Galois structure associated to the permutation subgroup $N = \langle(\overline{1_G}, \overline{\sigma}, \overline{\sigma^2}, \overline{\sigma^3})\rangle$.

8. Let E/K be a Galois extension with group of the form $J \times G'$ and call $L = E^{G'}$. Prove that the classical Galois structure on E/K is the induced Hopf-Galois structure from the classical Galois structures on E/L and L/K.