Natural homomorphism of Witt rings of a certain cubic order

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Witt ring of a commutative ring

R a commutative ring with 1.

M a left R-module.

$M^* = \mathrm{Hom}(M, R)$.

$\beta: M \times M \to R$ a **symmetric bilinear form** i.e. for every $x, y, z \in M$ and every $a, b \in R$:

$$\beta(ax + by, z) = a\beta(x, z) + b\beta(y, z), \qquad \beta(x, ay + bz) = a\beta(x, y) + b\beta(x, z),$$

and

$$\beta(x, y) = \beta(y, x).$$

Pair (M, β) is said to be a **bilinear module**.

Homomorphism $\hat{\beta}: M \to M^*$, $\hat{\beta}(m)(n) = \beta(m, n)$ is said to be the **adjoint homomorphism**.

Bilinear module (M, β) is said to be **nonsingular**, if the adjoint homomorphism $\hat{\beta}$ is an isomorphism.

Nonsingular bilinear module is said to be **inner product module**.

If M is a finitely generated projective module, then bilinear module (M, β) is said to be **bilinear space.**

Nonsinguilar bilinear space (M, β) is said to be **inner product space.**

Orthogonal sum $(M \perp N, \beta \perp \gamma)$ and **tensor prodcut** $(M \otimes N, \beta \otimes \gamma)$ of inner product spaces (M, β) and (N, γ) are defined as below:

$$(\beta \perp \gamma)(x \oplus y, t \oplus z) = \beta(x, z) + \gamma(y, t), \qquad (\beta \otimes \gamma)(x \otimes y, t \otimes z) = \beta(x, z)\gamma(y, t),$$

on simple tensors $x \otimes y, t \otimes z \in M \otimes N$.

By the universal property of tensor product it follows that $\beta \otimes \gamma$ uniquely extends to bilinear form $\beta \otimes \gamma: M \otimes N \times M \otimes N \to R$.

Proposition 1. *For bilinear modules* (M, β) *and* (N, γ)*,* $(M, \beta) \perp (N, \gamma)$ *is nonsingular if and only if* (M, β) *and* (N, γ) *are nonsinular.*

Proposition 2. *For bilinear modules* (M, β) *and* (N, γ)*, if* (M, β) *and* (N, γ) *are nonsingular then* $(M, \beta) \otimes (N, \gamma)$ *is nonsingular.*

Let (S, β) be a bilinear module and $N < S$.

Orthogonal complement of N in S is defined by:

$$N^{\perp} = \{s \in S \colon \beta(s, N) = 0\}$$

Inner product space (S, β) is said to be **metabolic**, if $S = M \oplus N$ and $N = N^{\perp}$.

Proposition 3. *Let* (S, α) *and* (T, β) *be metabolic spaces. Then* $(S \oplus T, \alpha \oplus \beta)$ *is a metabolic space.*

Proposition 4. *Let* (S, α) *be a metabolic space and* (X, β) *any inner product space. Then* $(S \otimes X, \alpha \otimes \beta)$ *is a metabolic space.*

Proposition 5. *Let* (M, β) *be an inner product space. Then* $(M, \beta) \perp (M, -\beta)$ *is a metabolic space.*

Two inner product spaces (M, β) and (N, γ) are said to be **isometric**, written $M \cong N$, if there exists a module isomorphism $f: M \to N$ satisfying

$$\gamma(f(m), f(m')) = \beta(m, m').$$

Proposition 6. *The isometry relation is an equivalence relation. It preserves orthogonal sums and tensor products, that is, for inner product spaces* M, N, M', N'*, if* $M \cong M'$ *and* $N \cong N'$*, then* $M \perp N \cong M' \perp N'$ *and* $M \otimes N \cong M' \otimes N'$*.*

Two inner product spaces M and N are said to be **similar**, written $M \sim N$, if there exist metabolic spaces S and T such that $M \perp S \cong N \perp T$.

Proposition 7. *The similarity relation is an equivalence relation. It preserves orthogonal sums and tensor products, that is, for inner product spaces* M, N, M', N'*, if* $M \sim M'$ *and* $N \sim N'$*, then* $M \perp N \sim M' \perp N'$ *and* $M \otimes N \sim M' \otimes N'$*.*

Denote by $W(R)$ the set of all similarity classes (also called Witt classes) of inner product spaces over an commutative ring R.

The class containing the space (M, β) is denoted by $\langle M, \beta\rangle$ or $\langle M\rangle$.

Addition and multiplication of classes are defined by

$$\langle M\rangle + \langle N\rangle = \langle M \perp N\rangle, \qquad \langle M\rangle \cdot \langle N\rangle = \langle M \otimes N\rangle.$$

Theorem 8. *The set $W(R)$ with addition and multiplication defined as above is a commutative ring with 1 called the **Witt ring of** R.*

Examples:

- $W(\mathbb{C})=\mathbb{Z}_2$ and $W(K)=\mathbb{Z}_2$, where K is an algebraically closed field.
- $W(\mathbb{R})=\mathbb{Z}$ and $W(R)=\mathbb{Z}$, where R is a real closed field.
- $W(\mathbb{Z}_p)=\mathbb{Z}_4$ if $p\equiv 3 \pmod 4$ and $W(\mathbb{Z}_p)=\mathbb{Z}_2[\mathbb{Z}_p^{\times}/\mathbb{Z}_p^{\times 2}]$, if $p\equiv 1 \pmod 4$.
- $W(\mathbb{Z})=\mathbb{Z}$.
- $W(k[X])=W(k)$, where k is a field, $\operatorname{char} k\neq 2$.

Witt functor

Let $f: R \to R'$ be a ring homomorphism.

R' is an R-module with multiplication:

$$R \times R' \to R', \qquad (a, a') \mapsto f(a) \cdot a'.$$

Let M be R-module.

Let $M' = R' \otimes_R M$.

M' is an R'-module with multiplication:

$$a' \cdot (b' \otimes m) = a'b' \otimes m.$$

Let $\beta : M \times M \to R$ be a bilinear form.

In view of the universal property of tensor product there is exactly one bilinear functional $\beta' : M' \times M' \to R'$ satisfying the condition

$$\beta'(a' \otimes m, b' \otimes n) = a'b' f(\beta(m,n)), \qquad a', b' \in R', m, n \in M.$$

The space (M', β') will be denoted by $f_{\#}(M, \beta)$ or simply $f_{\#}(M)$.

Proposition 9. *If* (M, β) *is an inner product space over* R*, then* $f_{\#}(M, \beta)$ *is an inner product space over* R'*.*

Ring homomorphism $f: R \to R'$ induces in this way Witt ring homomorphism $W(R) \to W(R')$ by

$$\langle M \rangle \mapsto \langle f_{\#}(M) \rangle.$$

This homomorphism is called the **natural homomorphism** of Witt ring and denoted by $f_{\#}$.

Thus we defined a covariant functor from the category of commutative rings with 1 to the category of Witt rings of rings:

$R \mapsto WR, \qquad (f: R \to R') \mapsto (f_{\#}: WR \to WR')$.

It will be called the **Witt functor**.

Theorem 10. *Let K be a number field. Let $\mathcal{O}_K$ be the ring of integers of K. Then $f_\# : W\mathcal{O}_K \to WK$ is injective.*

Theorem 11. *Let R be a Dedekind domain and K its field of fractions. Then $f_\# : WR \to WK$ is injective.*

M. Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 1969/1970, pp. 93–157.

Theorem 12. *Let A be a regular domain of dimension 2 and K its field of fractions. Then $f_\# : WA \to WK$ is injective.*

M. Ojanguren, *A splitting theorem for quadratic forms*, Commentarii mathematici Helvetici (1982), Volume: 57, page 145-157.

W. Pardon, *A Gersten conjecture for Witt groups*, In: Algebraic K-theory, Evanston 1976 Lecture Notes in Math. 551. Berlin, Heidelberg, New York: Springer 1984, pp. 261-328

What about other rings, whose field of fractions is equal to a number field...?

Orders

Let R be a Dedekind domain.

The ring $\mathcal{O} < R$, such that

- $\mathcal{O}$ is a noetherian ring,
- $\dim \mathcal{O} = 1$ (every nonzero prime ideal is maximal),
- R is the integral closure of $\mathcal{O}$ in the field of fractions of $\mathcal{O}$,
- R is a finitely generated $\mathcal{O}$-module,

is called an **order** in the ring R.

The Dedekind domain R is an order and we call it an maximal order.

Since orders are not necessarily integrally closed they are not Dedekind domains, and we lose the power of unique factorization of ideals.

Witt functors of orders

- $K=\mathbb{Q}(i)$, $\mathcal{O}=\mathbb{Z}[3i]$.Then $f_{\#}:W\mathcal{O}\to WK$ is not injective.

  Th. Craven, A. Rosenberg, R. Ware, *The Map of the Witt Ring of a Domain into the Witt Ring of its Field of Fractions*, Proceedings of the American Mathematical Society Vol. 51, No. 1 (Aug., 1975), pp. 25-30

- K – algebraic number field, $\mathcal{O}_K$ – its maximal order, $\mathcal{O}$ – an order such that the **conductor** $\mathfrak{f}=\{a\in\mathcal{O}_K|\,a\mathcal{O}_K\subseteq\mathcal{O}\}$ is even, i.e. $\mathfrak{f}\subseteq 2\mathcal{O}_K$. Then $f_{\#}:W\mathcal{O}\to WK$ is not injective.

  M. Ciemała, K. Szymiczek, *On injectivity of natural homomorphisms of Witt rings*, Annales Mathematicae Silesianae ([Nr] 21 (2007), s. 15-30).

- $K=\mathbb{Q}[\sqrt{d}]$, $d\not\equiv 1(\mathrm{mod}\,4)$, $\mathcal{O}=Z(f\sqrt{d})$. Let $2\nmid f$ and $f|\,d$. $f_{\#}:W\mathcal{O}\to WK$ is injective.

  B. Rothkegel, *Witt functor of a quadratic order*'', Annales Mathematicae Silesianae ([Nr] 29 (2015), s. 22-38)

Cubic orders

Theorem 13. (P.G. & M.P.) *Let* $K = \mathbb{Q}(\sqrt[3]{6})$, $\mathcal{O} = \mathbb{Z}[3\sqrt[3]{6}]$. *Then* $f_{\#} : W\mathcal{O} \to WK$ *is injective.*

Main tool

Proposition 14. *Let P be a Noetherian domain of dimension one with a finite singular locus. If $W(P_\mathfrak{p} \rightarrowtail \operatorname{int.cl.} P_\mathfrak{p})$ is a monomorphism for every prime $\mathfrak{p}$ of P, then $W(P \rightarrowtail \operatorname{int.cl.} P)$ is a monomorphism.*

P. Koprowski, *Witt morphisms*, Wydawnictwo Uniwersytetu Śląskiego, Katowice, 2012.

Main steps:

- We have to consider all localizations relative to prime ideals to apply Koprowski's Proposition. Then the prime ideals in the order extend explicitly to the prime ideals in the maximal order.
- Lets consider the rational prime 3. We are interested in orders $\mathcal{O}$ whose conductor is equal to $(3) \triangleleft \mathcal{O}$.
- Then lets consider the factorization of $(3) \triangleleft \mathcal{O}$ into prime ideals of $\mathcal{O}$. $\mathfrak{P} = \left(3, \sqrt[3]{6}\right)$ is the only prime ideal over (3) in $\mathcal{O}_K$.
- Further, we need to consider two cases, when the prime ideal over a given ideal is coprime with a conductor - and when it is not. One of these cases is trivial, and the other comes out in the calculations.
- The details of these computations are rather technical and will be skipped here.