# List 2: Hopf algebras

1. Let $G$ be a group. Prove that the $R$-group algebra $R[G]$ is an $R$-Hopf algebra.

2. Let $H$ and $H'$ be $R$-Hopf algebras. Prove that $H \otimes H'$ is an $R$-Hopf algebra with the following operations:

   - Multiplication map: $m_{H\otimes H'}\colon (H\otimes H')\otimes(H\otimes H') \longrightarrow H\otimes H'$, $m_{H\otimes H'}((a\otimes b)\otimes(c\otimes d)) = m_H(a\otimes c)\otimes m_{H'}(b\otimes d)$.
   - Unit map: $u_{H\otimes H'}\colon R \longrightarrow H\otimes H'$, $u_{H\otimes H'}(r) = r1_H \otimes 1_{H'}$.
   - Comultiplication map: $\Delta_{H\otimes H'} = (\mathrm{Id_H}\otimes \tau \otimes \mathrm{Id}_{H'})\circ(\Delta_H \otimes \Delta_{H'})\colon H\otimes H' \longrightarrow (H\otimes H')\otimes(H\otimes H')$, where $\tau\colon H\otimes H' \longrightarrow H'\otimes H$ is defined by $\tau(a\otimes b) = b\otimes a$.
   - Counit map: $\varepsilon_{H\otimes H'}\colon H\otimes H' \longrightarrow R$, $\varepsilon_{H\otimes H'}(a\otimes b) = \varepsilon_H(a)\varepsilon_{H'}(b)$.
   - Coinverse map: $S_{H\otimes H'}\colon H\otimes H' \longrightarrow H\otimes H'$, $S_{H\otimes H'}(a\otimes b) = S_H(a)\otimes S_{H'}(b)$.

3. Let $G$ and $H$ be finite groups. Prove that $R[G\times H]$ and $R[G]\otimes R[H]$ are isomorphic as $R$-Hopf algebras.

4. Let $A$ be an $R$-algebra and let $C$ be an $R$-coalgebra. Given $f, g \in \mathrm{Hom}_R(C, A)$, the **convolution** of $f$ and $g$ is defined as

   $$f * g \coloneqq m_A \circ (f\otimes g)\circ \Delta_C.$$

   Prove that $(\mathrm{Hom}_R(C, A), *)$ is a monoid (that is, it is associative and admits an identity element).

   **Hint:** It may help write down the definition of $f * g$ in terms of the Sweedler notation for the comultiplication.

5. Let $H$ be an $R$-Hopf algebra. Prove that the antipode $S_H$ is an anti-homomorphism of $R$-algebras, that is, $S_H(ab) = S_H(b)S_H(a)$ for all $a, b \in H$ and $S_H(1_H) = 1_H$.

   **Hint**: Use the uniqueness of the inverse of $m_H$, regarded as an element of the monoid $\mathrm{Hom}_R(H\otimes H, H)$ with the convolution.

6. Let $H$ and $H'$ be $R$-Hopf algebras, and let $f\colon H \longrightarrow H'$ be a homomorphism of $R$-bialgebras. Prove that $f$ is a homomorphism of $R$-Hopf algebras.

   **Hint**: Use the uniqueness of the inverse of $f$, regarded as an element of the monoid $\mathrm{Hom}_R(H, H')$ with the convolution.

7. Let $f\colon H \longrightarrow H'$ be a homomorphism of $R$-Hopf algebras.

   (a) Prove that $f(G(H)) \subseteq G(H')$.

   (b) Show that $|f(G(H))|$ divides $\gcd(|G(H)|, |G(H')|)$.

8. Let $H$ be a finite $R$-Hopf algebra.

   (a) Show that $H$ is a left $H$-module with the multiplication $m_H\colon H\otimes H \longrightarrow H$ as action. Write down the induced right $H^*$-comodule structure for $H$.

   (b) Show that $H^*$ is a right $H^*$-comodule with the comultiplication $\Delta_{H^*}\colon H^* \longrightarrow H^*\otimes H^*$ as coaction. Write down the induced left $H$-module structure for $H^*$.

9. Let $H$ be a finite $R$-Hopf algebra and let $A$ be a left $H$-module algebra. Let $\{h_i, f_i\}_{i=1}^n$ be a projective coordinate system for $H$ and let $\Psi\colon \mathrm{Hom}_R(H, A) \longrightarrow A \otimes H^*$ be the map defined by

$$\Psi(\varphi) = \sum_{i=1}^{n} \varphi(h_i) \otimes f_i, \quad \varphi \in \mathrm{Hom}_R(H, A).$$

Endow $\mathrm{Hom}_R(H, A)$ with the convolution product from Exercise 4. Prove that for every $f, g \in \mathrm{Hom}_R(H, A)$,

$$\Psi(\varphi * \psi) = \Psi(\varphi)\Psi(\psi).$$

**Hint:** Let $\Phi\colon A \otimes H^* \longrightarrow \mathrm{Hom}_R(H, A)$ be the inverse of $\Phi$. Try to prove that $\varphi * \psi = \Phi(\Psi(\varphi)\Psi(\psi))$.