List 1: Galois theory

1. Let K be a field with $\mathrm{char}(K) = 0$. Let L and M be finite extensions of K and M/K is Galois.

   (a) Prove that LM/L is Galois and that there is an embedding $\mathrm{Gal}(LM/L) \hookrightarrow \mathrm{Gal}(M/K)$, which becomes an isomorphism if $L \cap M = K$.

   (b) Suppose that L/K is also Galois. Show that LM/K is Galois and that there is an embedding $\mathrm{Gal}(LM/K) \hookrightarrow \mathrm{Gal}(L/K) \times \mathrm{Gal}(M/K)$, which becomes an isomorphism if $L \cap M = K$.

2. Let L be the splitting field of the polynomial $f(x) = x^4 + 6x^2 - 3$ over $\mathbb{Q}$. Determine completely the lattice of intermediate fields of $L/\mathbb{Q}$ and the lattice of subgroups of $\mathrm{Gal}(L/\mathbb{Q})$.

   Note: L is also the splitting field of the polynomial $x^4 - 3x^2 + 3$ over $\mathbb{Q}$.

3. Let L/K be a Galois extension with group G.

   (a) Show that G endowed with the Krull topology is a topological group.

   (b) Prove that the Krull topology on G is discrete if and only if L/K is finite. Deduce that the fundamental theorem of Galois theory at the infinite case is a generalization of the one for the finite case.

4. For each $m \in \mathbb{Z}_{>0}$, write L_m for the m-th cyclotomic field; that is, $L_m := \mathbb{Q}(\zeta_m)$, where ζ_m is a primitive m-th root of unity. In addition, for a prime number p, let $L_{p^\infty} = \bigcup_{n \in \mathbb{Z}_{>0}} L_{p^n}$ be the union of all the fields L_{p^n} (which is a field because $L_{p^n} \subset L_{p^{n+1}}$ for all $n \in \mathbb{Z}_{>0}$).

   (a) Prove that $L_m/\mathbb{Q}$ is Galois and that $\mathrm{Gal}(L_m/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$.

   Note: You do not need to prove the result that all the conjugates of ζ_m are ζ_m^k for $1 \leq k \leq m$ and $\gcd(k, m) = 1$.

   (b) Show that for each intermediate field E of $L_{p^\infty}/\mathbb{Q}$ such that $E/\mathbb{Q}$ is finite, there is some $n \in \mathbb{Z}_{>0}$ such that $E \subseteq L_{p^n}$. Deduce that if in addition $E/\mathbb{Q}$ is Galois, then it is abelian.

   (c) Prove that $L_{p^\infty}/\mathbb{Q}$ is Galois and that $\mathrm{Gal}(L_{p^\infty}/\mathbb{Q}) \cong (\mathbb{Z}_p)^\times$, the multiplicative group of the ring of p-adic integers.

   Note: You are allowed to use the definition of $\mathbb{Z}_p$ as a projective limit.