List 5: Hopf-Galois module theory

1. Let n be a square-free integer and let $L = \mathbb{Q}(\sqrt{n})$.

   (a) Find a necessary and sufficient condition for L to possess a normal integral basis. Prove its validity.

   (b) Justify that $\mathcal{O}_L$ is $\mathfrak{A}_{L/\mathbb{Q}}$-free.

2. Let L/K be a Galois extension of p-adic fields with group G. Prove that L/K is tamely ramified if and only if $\mathcal{O}_K[G] = \mathfrak{A}_{L/K}$.

3. Let $L = \mathbb{Q}(\sqrt[3]{2})$ and let H be the only Hopf-Galois structure on $L/\mathbb{Q}$.

   (a) Let $\alpha \in L - \mathbb{Q}$ and let $f(x) = x^3 + a_1x^2 + a_2x + a_3$ be its minimal polynomial over $\mathbb{Q}$. Prove that α is an H-normal basis generator for L if and only if $a_1 \neq 0$ and $a_1^2 \neq 3a_2$.
   Hint: Let $(\alpha_i)_{i=1}^3$ be the roots of f with $\alpha_1 = \alpha$. Use the symmetric identities of the roots to prove that $\sum_{i=1}^3 \alpha_i^2 = a_1^2 - 2a_2$.

   (b) Determine explicitly $\int_H^l$. Is H unimodular?
   Hint: Find a generator w of H as a K-algebra and work with the K-basis $\{\mathrm{Id}, w, w^2\}$ of H. When considering products, use a degree 3 identity satisfied by w.

4. Let L/K be a (G, G')-separable H-Galois extension of p-adic fields with normal closure $\widetilde{L}$ and let N be the regular and G-stable subgroup of $\mathrm{Perm}(G/G')$ such that $H = \widetilde{L}[N]^G$. Prove that $\mathcal{O}_{\widetilde{L}}[N]^G \subseteq \mathfrak{A}_H$.

5. Let K be a p-adic field with valuation ring $\mathcal{O}_K$ and let G be a cyclic group with generator σ. Call $f = \sigma - 1_G$. Prove that $\mathcal{O}_K[f]$ is an $\mathcal{O}_K$-Hopf order in $K[G]$[1].

6. For the following extensions of number or p-adic fields L/K and Hopf-Galois structure H on L/K, determine whether $\mathcal{O}_L$ is $\mathfrak{A}_H$-free or not. In the case that L/K admits a unique Hopf-Galois structure, H is not specified. You are allowed to use any information at the LMFDB database.

   (a) $L = \mathbb{Q}(\sqrt[3]{2})$ and $K = \mathbb{Q}$; Number field 3.1.108.1.

   (b) $L = \mathbb{Q}(\sqrt{3}, \sqrt{2})$, $K = \mathbb{Q}$ and $H = H_3$ with the notation of Theorem 3.6.20; Number field 4.4.2304.1.

   (c) $L = \mathbb{Q}_5(\alpha)$ and $K = \mathbb{Q}_5$, where $\alpha^3 + 5 = 0$; p-adic field 5.1.3.2a1.1.

   (d) $L = \mathbb{Q}_5(\alpha)$ and $K = \mathbb{Q}_3$, where $\alpha^6 + 6\alpha^2 + 3 = 0$; p-adic field 3.1.6.7a2.1.

[1] In fact, the $\mathcal{O}_K$-Hopf orders of $K[G]$ are $\mathcal{O}_K[\pi_K^{-i} f]$ for $0 \leq i \leq \lfloor \frac{e}{p-1} \rfloor$.