Summary

In this course, we present a constructive introduction to Hopf-Galois theory, a generalization of Galois theory with the use of Hopf algebras, and its use to broaden the domain of Galois module theory.

Galois theory arises from the remark that the symmetries of the solutions of an algebraic equation form a group. In the modern language, the Galois group G of a Galois extension of fields L/K permutes the roots of a polynomial $f \in K[X]$ as long as they belong to L, and therefore can be embedded in the permutation group S_n. This action of G on L, by automorphisms, extends naturally to an action of the Galois group algebra $K[G]$ on L by endomorphisms. The advantage of this approach is that $K[G]$ can be endowed with K-Hopf algebra structure, richer than the one of a group, and permits to characterize the Galois condition for a finite and separable extension of fields.

In this setting, the notion of Hopf-Galois extension appears naturally: a finite extension of fields L/K which receives the action of a K-Hopf algebra H imitating the properties of the action of $K[G]$ on L. The pair formed by such a Hopf algebra and its action on L is called a Hopf-Galois structure. Hopf-Galois theory studies Hopf-Galois structures and the algebraic information they codify on the extension to which they refer.

Among the very basics of Hopf-Galois theory, there's the Greither-Pareigis theorem: the Hopf-Galois structures on a finite and separable degree n extension L/K are in bijective correspondence with a subclass of permutation subgroups of S_n depending only on the Galois groups of $\widetilde{L}/K$ and $\widetilde{L}/L$, where $\widetilde{L}$ is the normal closure of L/K. This groundbreaking result linked the study of Hopf-Galois extensions and their Hopf-Galois structures with group theory and gave rise to countless consequences.

The use of Hopf-Galois theory allows to generalize the picture corresponding to Galois module theory, which studies the structure of the ring of integers in a Galois extension of global or local fields as a module over an object depending on its Galois group. Under this perspective, for a Hopf-Galois extension, one can consider the module structure of the ring of integers over an object in a Hopf-Galois structure on the extension. This extends the possibilities of Galois theoretic problems in number theory to this more general setting.

Syllabus

Chapter 1: Preliminaries on Galois theory and Hopf algebras

1.1 Field theory and Galois theory.

1.2 Hopf algebras and their actions on modules.

Chapter 2: Hopf-Galois theory and the Greither-Pareigis correspondence

2.1 Hopf-Galois extensions and Hopf-Galois objects. The Hopf-Galois correspondence.

2.2 Hopf-Galois structures on separable extensions. The Greither-Pareigis theorem. Almost classically Galois extensions.

2.3 Byott's translation. Number of Hopf-Galois structures on a separable extension.

2.4 Revisiting the Greither-Pareigis correspondence. Explicit form. The Galois case.

2.5 Applications of Greither-Pareigis theorem. Induced Hopf-Galois structures. Opposite group. Fixed-point-free abelian endomorphisms.

Chapter 3: Hopf-Galois module theory

3.1 Review on algebraic number theory: extensions of global and local fields and their rings of integers. Ramification theory.

3.2 Normal basis theorem for Galois and Hopf-Galois extensions. Normal integral bases and Noether's theorem. Associated orders. The Hopf-Galois case.

3.3 Further elements of Hopf-Galois module theory: H-tame extensions, maximal orders, Hopf orders.

3.4 The ring of integers in extensions of p-adic fields. Weak and strong congruences for ramification breaks. Hopf orders as associated orders. Scaffolds.

3.5 The ring of integers in extensions of number fields. Local and global freeness. The reduction method. Criteria for freeness over induced Hopf-Galois structures.

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