Chapter 2

Hopf-Galois theory and the Greither-Pareigis correspondence

1 Hopf-Galois extensions and Hopf-Galois objects

In this section we will introduce Hopf-Galois structures from two viewpoints: via module algebras, and via comodule algebras. Given a Hopf-Galois structure, there is a method of turning sub-Hopf algebras (quotient Hopf algebras respectively) into subalgebras of the algebra which carries a Hopf-Galois structure. This is in a way a generalization of the classical correspondence in Galois theory of fields, but it is in a sense weaker, as not all subalgebras are reached by this process in general. We will soon describe this method, but for a proof of some main properties we will need a better understanding of algebras (via Γ-sets), an so some arguments have to be postponed

Let K be any base field. All algebras over K are assumed finite-dimensional over K unless said otherwise; the algebras bearing a Hopf-Galois structure will be assumed to be commutative. Hom groups and tensor products without subscript are taken over K.

Let H be a K-Hopf algebra. Recall that the defining map $\alpha_A : H \otimes A \longrightarrow A$ of a module algebra A makes H act on A, by the simple rule $h \cdot x = \alpha_A(h \otimes x)$ for $h \in H, x \in A$. The defining map $\beta_A : A \longrightarrow A \otimes H^*$ looks as follows in Sweedler notation: $\beta_A(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}$, where $x \in A$, and the factors $x_{(0)}$ and $x_{(1)}$ indicate elements of A and H^* respectively (see (1.4)).

There are two standard types of canonical isomorphisms for any triple X, Y, Z of K-vector spaces:

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \text{ (Hom-Tensor adjunction)}$$

and

$$\mathrm{Hom}(X, Y \otimes Z) \cong \mathrm{Hom}(X, Y) \otimes Z.$$

This gives (recall that $H^* = \mathrm{Hom}(H, K)$ and $A = K \otimes A$):

$$\begin{aligned} \mathrm{Hom}(H \otimes A, A) &\cong \mathrm{Hom}(A, \mathrm{Hom}(H, A)) \\ &\cong \mathrm{Hom}(A, \mathrm{Hom}(H, K \otimes A)) \\ &\cong \mathrm{Hom}(A, \mathrm{Hom}(H, K) \otimes A) \\ &= \mathrm{Hom}(A, A \otimes H^*). \end{aligned}$$

The twist in the last step is necessary, not for the existence of the isomorphism, but to make it behave, with respect to module and comodule structures.

Definition 2.1.1. *Let H be a K-Hopf algebra and A a left H-module algebra. Consider the map $j : A \otimes H \longrightarrow \mathrm{End}(A) = \mathrm{Hom}(A, A)$ defined by $j(x \otimes h)(y) = x \cdot h(y)$. In other words: $j(x \otimes h)$ is the action of h on A, followed by left multiplication with the element x. Then A is said to be an H-Hopf-Galois (or H-Galois) extension if the map j is bijective.*

We remark that if j is bijective and n, m denote the K-dimensions of A and H respectively, then we get an equality $nm = dim(A \otimes H) = dim(\mathrm{End}(A)) = n^2$ and hence $n = m$.

The prime example is the Hopf algebra $K[G]$, where G is any finite group, for any $g \in G$ we have $\Delta_{K[G]}(g) = g \otimes g$, the antipode $S_{K[G]}$ sends g to its inverse, and $\varepsilon_{K[G]}(g) = 1$. Assume L/K is G-Galois. Then L becomes an H-module algebra by defining $\alpha_L(g \otimes x) = g(x)$; the action of the Galois group is simply encoded as a map $K[G] \otimes L \longrightarrow L$. We check that L is indeed a module algebra: let $x, y \in L$ and $g \in G$. Then $g(xy) = g(x)g(y)$, and on the other hand

$$\Delta_{K[G]}(g)(x \otimes y) = (g \otimes g)(x \otimes y) = g(x) \otimes g(y),$$

which contracts to $g(x)g(y)$ under multiplication. The condition concerning the unit map is obviously satisfied.

Dedekind has already showed that the elements of G, considered as elements of $\mathrm{End}(L)$, are linearly independent, if we make $\mathrm{End}(L)$ into an L-vector space, vie left multiplication by elements of L. But this is exactly saying that the map j is injective. So for reasons of dimension, j is bijective.

Let us discuss H^* and the comodule-algebra structure $\beta_L : L \longrightarrow L \otimes H^*$ in detail, to get a clear picture in this classical setting. A basis for H^* is given by the elements e_g ($g \in G$), where $e_g : K[G] \longrightarrow K$ is extraction of the g-th coefficient: $e_g(\sum_{h \in G} r_h h) = r_g$. We calculate the structure maps. First, since every $k \in G$ satisfies $\Delta_{H^*}(k) = k \otimes k$, we get $(e_g \cdot e_h)(k) = e_g(k)e_h(k)$ for all $g, h, k \in G$; this is 1 if $g = h = k$ and 0 otherwise. Therefore $e_g e_h$ is e_g if $g = h$ and 0 otherwise. Elements e with $e^2 = e$ are commonly called idempotents.

Now for the diagonal map of the dual; it is given by $\Delta_{H^*}(e_g)(h \otimes k) = e_g(hk)$. This is 1 if $hk = g$ and 0 otherwise, so $\Delta_{H^*}(e_g)$ is the sum of all $e_h \otimes e_k$ such that $hk = g$. We leave it to the readers to determine the augmentation and the antipode of H^*.

The dual H^* can be described more simply as the set of maps $\mathrm{Maps}(G, K)$, also written K^G; a G-tuple $(r_g)_{g \in G}$ is simply the map on G sending g to r_g. In other terms, the tuple $(r_g)_{g \in G}$ is $\sum_g r_g e_g$, and the idempotent e_g corresponds to the tuple having exactly one 1 at position g and zeros otherwise. From this one also sees that $L \otimes H^*$ likewise identifies with L^G (the set of maps from G to L). We may now elucidate the comodule structure.

The general rule for getting β_A from α_A uses a "dual basis" $\{h_i, \phi_i\}_i$ (see Definition 1.2.24) for the pair (H, H^*), and says $\beta(x) = \sum_i m(h_i \otimes x) \otimes \phi_i = \sum_i h_i(x) \otimes \phi_i$. (Recall that the rule going the other way is even simpler). In our case we already have a beautiful dual basis: the elements $g \in G$ for H, and the idempotents e_g for H^*. Thus:

$$\beta(x) = \sum_{g \in G} g(x) \otimes e_g.$$

If we look at the identification $L \otimes K^G = L^G$, the last sum is simply the map $G \longrightarrow L$ taking the value $g(x)$ at g; in other words, the tuple $(g(x))_{g\in G}$.

We need another definition.

Definition 2.1.2. *Let J be another K-Hopf algebra, and A be a J-comodule algebra via $\beta = \beta_A : A \longrightarrow A \otimes J$. We define a map $\gamma : A \otimes A \longrightarrow A \otimes J$ via $\gamma(x \otimes y) = (x \otimes 1)\beta(y)$. (So it is identity on the lefthand tensor factor, and restricted to the righthand tensor factor of its source, it is β.) Then A is called a right H-object if the map γ is bijective.*

Let us show that in the above example, the map $L \longrightarrow L \otimes H^* = L^G$ gives an H^*-Galois object. Let $\{x_1, \ldots, x_n\}$ be a K-basis of L. Injectivity of $\gamma : L \otimes L \longrightarrow L^G$ means that the elements $\beta(x_i)$ are not only K-linearly independent, but even over L. Let us show this. We need that the n row vectors $(g(x_i))_g$ are L-linearly independent. It is equivalent to say that the square matrix $M = \big(g(x_i)\big)_{i,g}$ has maximal rank. But now we look at the columns $\big(g(x_i)\big)_i$ of M. They are L-independent iff the elements g of G are L-independent considered as maps $L \longrightarrow L$. And this is known, again thanks to Dedekind.

Before proceeding, let us present another important class of Hopf-Galois extensions/objects.

Definition 2.1.3. *Let n be a fixed positive integer; a K-algebra A is called **fully n-graded** if*

$$A = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} A_i, \quad \dim_K(A_i) = 1 \quad \forall i$$

and for all $i, j \in \mathbb{Z}/n\mathbb{Z}$, the multiplication of A induces an isomorphism $A_i \otimes A_j \longrightarrow A_{i+j}$. In simpler terms, if $A_i = Kx_i$, then $x_i x_j = u_{i,j} x_{i+j}$ where $u_{i+j} \in K$ is not zero.

Example 2.1.4. Assume $u \in K$, α is a root of $x^n - u$, and the latter polynomial is irreducible. Put $A = K(\alpha)$ (a field), and $A_i = K\alpha^i$.

Now let C be another cyclic group of order n, written multiplicatively, with generator c. We will show that any fully n-graded algebra A is an H-Galois extension with $H = K^C$ and an H^*-Galois object with $H^* = (K^C)^* = K[C]$. Let us begin with the latter. The map $\beta : A \longrightarrow A \otimes H^* = A[C]$ is defined as follows: Put $\beta x = x \otimes c^i$ if $x \in A_i$ (one says: x is homogeneous of degree i), and extend by linearity. Coassociativity is easy: take $x \in A_i$. Then $(1 \otimes \Delta)\beta(x) = x \otimes c^i \otimes c^i$, and $\beta \otimes 1$ applied to $\beta(x) = x \otimes c^i$ gives the same. Let us also check that the induced map γ is bijective. Take a basis x_i of every A_i. Then γ maps $x_j \otimes x_i$ to $x_j x_i \otimes c^i$, and the "fully graded" condition ensures that these elements generate all of $A[C]$. This makes γ surjective, hence bijective.

Let us quickly describe the corresponding H-Galois structure on the fully n-graded algebra A; details left to reader. Recall that $H = K^C$ has a K-basis $(e_0, e_1, \ldots e_{n-1})$ of idempotents, each e_i acting on $K[C]$ as extraction of the coefficient at c^i. One can then check that $e_i \in H$ acts on A as projection to the direct summand A_i. – We note in passing that one can prove a converse: indeed A is an H^*-Galois object (or as we will see: equivalently, an H-Galois extension) only if A is fully graded and the structures arise exactly as described.

We will now show that our definitions of Hopf-Galois extension/object behave well in general when we switch the side. In the concrete examples above, we checked it or at least mentioned it.

Proposition 2.1.5. *Let H be a K-Hopf algebra, and $\alpha : H \otimes A \longrightarrow A$, $\beta : A \longrightarrow A \otimes H^*$ be (co)module algebra structures that correspond to each other. Then A is an H-Galois extension if and only if A is an H^*-Galois object.*

Proof. The only real point is that the map j (attached to α) is bijective if and only if the map γ (attached to β) is bijective. Ensuring this equivalence is a bit technical, and we omit some details. Recall that the algebra A is assumed to be commutative.

We start by exhibiting two canonical K-linear maps. Both are isomorphisms; we will not check this (it can be done by picking bases for example). They are:

$$\eta : A \otimes H \longrightarrow \mathrm{Hom}_A(A \otimes H^*, A), \quad \eta(a \otimes h)(b \otimes \phi) = \phi(h) \cdot ab,$$

and

$$\delta : \mathrm{Hom}_K(A, A) = \mathrm{End}(A) \longrightarrow \mathrm{Hom}_A(A \otimes A, A), \quad \delta(f)(a \otimes b) = af(b).$$

Recall our two maps $j : A \otimes H \longrightarrow \mathrm{End}(A)$ and $\gamma : A \otimes A \longrightarrow A \otimes H^*$, given by $j(a \otimes h)(b) = ah(b)$ and $\gamma(a \otimes b) = (a \otimes 1) \cdot \beta(b)$. The map γ gives rise to another map $\gamma^* = \mathrm{Hom}_A(\gamma, A)$ going from $\mathrm{Hom}_A(A \otimes H^*, A)$ to $\mathrm{Hom}_A(A \otimes A, A)$. We consider the following diagram:

$$\begin{array}{ccc} A \otimes H & \xrightarrow{\quad j \quad} & \mathrm{End}(A) \\ \downarrow \eta & & \downarrow \delta \\ \mathrm{Hom}_A(A \otimes H^*, A) & \xrightarrow{\gamma^*} & \mathrm{Hom}_A(A \otimes A, A). \end{array}$$

If we can prove that this square commutes, then we are done: given that the vertical maps are bijective, the upper horizontal map will be bijective if and only if the lower one is.

As a preparation we calculate: $\gamma^*(f)(a \otimes b) = f(\gamma(a \otimes b)) = f((a \otimes 1) \cdot \beta(b)) = f(\sum_{(b)} ab_{(0)} \otimes b_{(1)})$. Now we take an element $a \otimes h$ in the upper left hand module and chase it two ways. We have $j(a \otimes h)(b) = ah(b)$, so

$$\delta j(a \otimes h)(c \otimes b) = c\, j(h \otimes a)(b) = ca\, h(b).$$

Now for the other way round the square (f being replaced by $\eta(a \otimes h)$):

$$\gamma^* \eta(a \otimes h)(c \otimes b) = \eta(a \otimes h)(\sum_{(b)} cb_{(0)} \otimes b_{(1)}) = a \sum_{(b)} cb_{(0)} \otimes h(b_{(1)}) = ac\, h(b).$$

This concludes the argument. □

Now we turn to a version of the classical Galois correspondence. For a G-Galois extension L/K, we can associate to every subgroup $U < G$ an intermediate field $\mathrm{Fix}(U) = \mathrm{Fix}(L, U) = \{x \in L : \sigma(x) = x \ \forall \sigma \in U\}$, and it is known that we obtain an inclusion-reversing bijection between the set (lattice) of all subgroups of G and the set (lattice) of all fields between K and L (see Theorem 1.1.51). In the Hopf setting, there will be two versions again, on the module side and on the comodule side. It will be important to see that these two ways of viewing the correspondence are equivalent. We say already here that in general the new correspondence will not

be perfect - we will not get all intermediate algebras between K and A, not even if $A = L$ is a field.

If L/K is G-Galois, it is a H-Galois extension with $H = K[G]$ as seen before. For any subgroup $U < G$ we have the sub-Hopf algebra $H' = K[U]$ in H, and the fixed field $E = \mathrm{Fix}(U)$ can be described as

$$E = \{x \in L : h(x) = \varepsilon(h)(x)\ \forall h \in H'\}.$$

In other words, E is the subalgebra annihilated by the augmentation kernel of the sub-Hopf algebra H'. This lends itself to a generalization. We note already here: If J and J' denote the duals of H and H' respectively, then $J = K^G$, $J' = K^U$, and the induced surjective homomorphism $J \longrightarrow J'$ of Hopf algebras, call it g, is simply restricting a G-tuple to an U-tuple. We will come back to this.

Definition 2.1.6. *Let A be an H-Galois extension, and $H' \subset H$ an arbitrary K-sub-Hopf algebra. The fixed algebra* $\mathrm{Fix}(A, H') = \mathrm{Fix}(H')$ *is defined as the set* $\{x \in A : h(x) = \varepsilon(h)(x)\ \forall h \in H'\}$. *Note that we use the simpler notation $h(x)$ instead of $\alpha_A(h \otimes x)$.*

It is obvious that $\mathrm{Fix}(H')$ is a subspace of A.

This construction reduces to the usual "fixed field" operation in the classical case, as seen above.

Example 2.1.7. Let us review the fully graded situation for another example. We take A to be a fully n-graded K-algebra, with its structure of H-Galois extension, where $H = K^C$, and C is cyclic of order n generated by c. If m is a divisor of n, and C' cyclic of order m, then there is a canonical surjective group homomorphism $C \longrightarrow C'$, mapping c to $\overline{c}$ (a generator of C'). This gives a sub-Hopf algebra $H' \subset H$, consisting of the tuples (r_i) whose i-entry $r_i \in K$ depends only on i modulo m, not just modulo n. We look at elements $a = \sum_i a_i \in A$, where $a_i \in A_i$, and we ask when such an element is annihilated by all $h - \varepsilon(h)$ with $h \in H'$. Let $0 \leq k < n$ not be divisible by m. Then there is an m-periodic tuple r having $r_0 = 0$ and $r_k = 1$. Applying it to a, we get zero only if $a_k = 0$. So we find that $\mathrm{Fix}(H')$ consists exactly of those a which have nothing in all degrees k that are not divisible by m; and this is the fully n/m-graded algebra $\sum_{0 \leq i < n; m|i} A_i = A_0 \oplus A_m \oplus A_{2m} \oplus \ldots$.

Let us now describe the Fix construction on the comodule side, starting with a motivating example. We will conclude this section by a proof that we get the same outcome of the Fix construction on both sides.

Consider $A = L$ a field Galois extension of K with group N. Then L is a J-object, with $J = K^N = \mathrm{Maps}(N, K)$; the map β sends $x \in L$ to the tuple $(\sigma(x))_{\sigma \in N}$. Let N' be any subgroup of N. This gives a surjective homomorphism $g : J \longrightarrow J' = K^{N'}$, simply by restricting tuples. We then have two maps $f_1, f_2 : L \longrightarrow L \otimes J = L^{N'}$. The first is β followed by $L \otimes g$, so x goes to $(\tau(x))_{\tau \in N'}$. The map f_2 sends $x \in L$ to $(x, \ldots, x)$, that is, the N'-tuple which has all entries equal to x.

Then it is pretty obvious that $f_1(x) = f_2(x)$ if and only if x is fixed under the subgroup N'; in other words, the so-called equalizer $\{x \in L : f_1(x) = f_2(x)\}$ of the two maps f_1 and f_2 is the fixed field of N' inside L. We now generalize this construction.

Let A be a Hopf-Galois object for the Hopf algebra J, and let $g : J \longrightarrow J'$ be any surjective homomorphism of K-Hopf algebras. Let $u = u_{J'}$ be the unit map of the

algebra J', that is, the map $K \longrightarrow J'$ that sends $r \in K$ to $r \cdot 1_{J'}$. (One might consider u as an inclusion, but in the example $J' = K^{N'}$ this would be a bit unnatural as we will see.) We define $\mathrm{Fix}(g) \subset A$ to be the equalizer of the two maps

$$\begin{aligned} A \longrightarrow A \otimes J \quad &\longrightarrow \quad A \otimes J', \quad x \longmapsto (id_A \otimes g)\beta(x); \\ A \longrightarrow A \otimes J \quad &\longrightarrow \quad A \otimes J', \quad x \longmapsto (id_A \otimes u\varepsilon)\beta(x). \end{aligned}$$

Let us check that this reproduces taking a fixed field, in the particular case just discussed: Here $g : K^N \longrightarrow K^{N'}$ is the restriction map. The first map in the display just above specializes to the map f_1. We look at $u\varepsilon$: As $u : K \longrightarrow K^{N'}$ is the diagonal, sending x to $(x, \ldots, x)$, we get that $u\varepsilon$: sends an N-tuple y to the N'-tuple all of whose entries are y_e (the e-entry of y). Hence the second map in the display specializes to f_2, as desired.

The proof of the following result has no particular difficulties (use the definitions) and is omitted.

Proposition 2.1.8. *1. If A is an H-Hopf-Galois extension and H' a sub-Hopf algebra of H, then the set* $\mathrm{Fix}(A)$ *is a subalgebra of A.*

2. If A is a J-Hopf-Galois object and $g : J \longrightarrow J'$ a surjection of Hopf algebras, then the set $\mathrm{Fix}(g)$ *is a subalgebra of A.*

The operators Fix enjoy more properties. They are injective in the sense that different sub-Hopf algebras (quotient Hopf-algebras) lead to different (co)fixed algebras, and one can also predict the dimension of the fixed algebra. To prove these statements, we need more technique, so this is deferred. For the moment, we "only" prove compatibility of the Fix operators on the two sides. We consider the usual situation: A is a H-Hopf-Galois extension via $\alpha : H \otimes A \longrightarrow A$, and the corresponding structure of A as an $H^* = J$-Galois object is $\beta : A \longrightarrow A \otimes J$. Let H' be a sub-Hopf algebra of H. Dualizing the inclusion $H' \to H$ gives a surjective Hopf algebra map $J \longrightarrow J' = (H')^*$, which will be denoted g.

Theorem 2.1.9. *With these notations and assumptions, the fixed algebra* $\mathrm{Fix}(H') \subset A$ *agrees with the cofixed algebra* $\mathrm{Fix}(g)$.

Proof. Recall the transition rule: if $\beta(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}$ with $x_{(1)} \in J$, then for $v \in H$, we have $u(x) = \sum_{(x)} x_{(0)} \cdot x_{(1)}(v)$. Let us assume $x \in \mathrm{Fix}(g)$, so $\sum_{(x)} x_{(0)} \otimes g(x_{(1)}) = \sum_{(x)} x_{(0)} \otimes u_J\varepsilon_J(x_{(1)})$, where the structural maps i_J, ε_J belong to J. Then $i_J(1)$ applied to $v \in H$ is the scalar $\varepsilon_H(v)$. We get for $v \in H'$ (the g may be inserted because v is not just in H but in H'):

$$\begin{aligned} v(x) &= \sum_{(x)} x_{(0)} \cdot x_{(1)}(v) \\ &= \sum_{(x)} x_{(0)} \cdot g(x_{(1)})(v) \\ &= \sum_{(x)} x_{(0)} \cdot i_J\varepsilon_J(x_{(1)})(v) \\ &= \sum_{(x)} x_{(0)} \cdot \varepsilon_H(v)\varepsilon_J(x_{(1)}) \\ &= \varepsilon_H(v) \cdot x, \end{aligned}$$

so x is indeed in $\mathrm{Fix}(H')$.

For the other direction, assume that x is in $\mathrm{Fix}(H')$. We choose dual bases (u_i, h_i) (with $i = 1, \ldots, n$) for H and J such that the following hold. h_1 is the unit element of J (that is, $h_1 = \varepsilon_H$); $u_1 = 1_H$; $u_1, \ldots, u_k$ are a basis of H' and all of them but u_1 are in the kernel of augmentation; and $h_{k+1}, \ldots, h_n$ are a basis of the kernel of $g : J \longrightarrow J'$. In particular, $(u_i, \overline{h_i})_{1 \le i \le k}$ is a dual basis for the pair H', J'. By the general transition rule from modules to comodules, we have $\beta(x) = \sum_{i=1}^{n} u_i(x) \otimes h_i$. Hence we obtain (denoting the map $g : J \longrightarrow H'$ simply by overbar)

$$(1 \otimes g)\beta(x) = \sum_{i=1}^{n} u_i(x) \otimes \overline{h_i}.$$

We now use that for $i > k$ the term $\overline{h_i}$ vanishes, that $u_1(x) = x$, and $u_i(x) = 0$ for $i = 2, \ldots k$ since x is H'-fixed; so the RHS in the preceding equation is simply $x \otimes \overline{h_1}$. On the other hand, $u_J \varepsilon_J$ annihilates all h_i with $i > 1$, so we likewise obtain $(1 \otimes u_J \varepsilon_J)(\sum_i u_i(x) \otimes h_i = 1 \cdot x \otimes u_J \varepsilon_J(h_1) = x \otimes h_1$. Therefore x is cofixed under g, as desired. □

2 Hopf-Galois structures on separable extensions

2.1 Describing (Hopf) algebras via Γ-sets

Our goal in this section is a description of finite-dimensional commutative algebras A over a fixed base field K by a simpler object, almost combinatorial in nature. A description of (finite-dimensional) commutative K-Hopf algebras will also emerge almost for free. This technique will allow to prove some missing facts about (co)fixed algebras in a Hopf-Galois situation, and it is an easy way towards Greither-Pareigis (GP) theory, which will be treated in the next section. We will assume for simplicity that our base field is of characteristic zero (or a finite field), so that all field extensions are separable. (It would be sufficient to assume that all algebras that we use are "separable", but then we would have to define what that means.)

Every field K has an algebraic closure $\overline{K}$, which can be thought of as a filtered union of finite (in particular algebraic) field extensions L/K. In every concrete situation it would be enough to work with one such extension L/K. But very often that field L needs to be changed (e.g. enlarged) in a longer argument, and it is a hindrance to fix such an L too early. The situation is similar to polynomials: one needs the full polynomial ring a priori, and bounds on degrees of polynomials often tend to obscure theoretical arguments that are otherwise clear. The price to pay is that $\Gamma = \Gamma_K$, the automorphism group of $\overline{K}/K$, is (almost always) infinite. But this group bears a very nice topology, called profinite. It suffices to know the following facts: The open subgroups U are exactly the fixed groups of finite extensions L/K, and they have finite index, equal to $[L : K]$, in Γ; every open subgroup contains another subgroup V still of finite index which is normal in Γ, and then $G = \Gamma/V$ is the Galois group of the fixed field $\mathrm{Fix}(V)/K$. The group Γ will act on various finite sets , and all actions will be continous in the following sense: for every $s \in S$, the so-called stabilizer $\Gamma_s = \{\gamma \in \Gamma : \gamma s = s\}$ is open. Then the intersection of all stabilizers is again open, contains an open normal subgroup V, and "in reality" the action is then via the finite group $G = \Gamma/V$.

After these preliminaries, let us repeat what a Γ-set S is: it is a set together with a map $\Gamma \times S \longrightarrow S$ denoted by a dot in the middle or by nothing, such that some obvious axioms are satisfied: $e_\Gamma s = s$, and $\beta(\gamma s) = (\beta\gamma)s$ for all $s \in S$, $\beta, \gamma \in \Gamma$. We also say: The group Γ operates on the set S. The stabilizer of an element has already be defined; it is always a subgroup. A typical example is the set $S = \{1, \ldots, n\}$, acted upon by the symmetric group of order $n!$.

Another example is the linear group $\mathrm{GL}(n, K)$ action (via left multiplication by matrices) on the column space K^n.

We offer some more remarks about group operations, for later use.

(1) The notion of morphism between two Γ-sets is so obvious that we do not have to write it down.

(2) If $s_0 \in S$, then $\Gamma s_0 = \{\gamma s : \gamma \in \Gamma\}$ is a Γ-subset of S, and it does not contain any nonempty smaller Γ-subset. Such subsets are called orbits. Every Γ-set S is the disjoint union of its orbits in an essentially unique way.

(3) For any subgroup $\Delta < \Gamma$, the set of cosets $\gamma\Delta$, $\gamma \in \Gamma$, is a Γ-set, via the operation $\rho(\gamma\Delta) = (\rho\gamma)\Delta$. It is written Γ/Δ (careful: this need not be a group unless Δ is normal), and it has only one orbit.

(4) Every orbit in a Γ-set is isomorphic to the Γ-set Γ/V, where V is defined to be the stabilizer of a chosen element.

Let $\mathcal{A}_K$ be the class (or category) of all commutative finite-dimensional K-algebras without nilpotent elements, and let $\mathcal{S}_\Gamma$ be the category of all finite Γ-sets (with continuous action, always), where Γ is short for Γ_K. Our goal is to establish inverse bijections (more precisely equivalences of categories) $\Phi : \mathcal{A}_K \longrightarrow \mathcal{S}_\Gamma$ and Ψ going the other way, and to see what happens to Hopf algebras under this correspondence. We need a minimum of algebraic informaton on algebras.

Proposition 2.2.1. *Let A be a finite-dimensional commutative K-algebra. If A has no nonzero nilpotent elements, then A is isomorphic to a finite product of fields L_i with $[L_i : K] < \infty$. (The reverse implication is also true, and obvious.)*

Proof. (a) We first argue that A has only finitely many maximal ideals. Indeed let $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ be an infinite list of distinct maximal ideals. If we take $x_i \in \mathfrak{m}_i \setminus \mathfrak{m}_{s+1}$ for all $i \leq s$, then the product $x_1 \cdots x_s$ is in the intersection $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$ but not in $\mathfrak{m}_{s+1}$. Hence the intersection $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_{s+1}$ is properly smaller than $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$, which means that we have a properly descending infinite chain of ideals, which is of course impossible.

(b) Every prime ideal $\mathfrak{p}$ of A is maximal. Indeed if $\mathfrak{p}$ is prime, the factor ring $A/\mathfrak{p}$ is still finite-dimensional over K and has no zero-divisors. It is well known that this forces $A/\mathfrak{p}$ to be a field. That is, the ideal $\mathfrak{p}$ was maximal.

(c) The set of nilpotent elements in A is equal to the intersection of all prime ideals. This is a standard fact with a standard proof, which will be omitted here.

(d) Now let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the complete list of the maximal ideals of A. This is also the list of all prime ideals, so the intersection of the $\mathfrak{m}_i$ is zero, by part (c)

and our hypothesis. By the Chinese Remainder Theorem we get $A \cong A/0 \cong \prod_{i=1}^{t} A/\mathfrak{m}_i$, and it suffices to put $L_i = A/\mathfrak{m}_i$.

□

We now define the map (functor) $\Phi : \mathcal{A}_K \longrightarrow \mathcal{S}_\Gamma$ by setting

$$\Phi(A) = \mathrm{Alg}_K(A, \overline{K}).$$

Here $\mathrm{Alg}_K(A, \overline{K})$ denotes the set of K-algebra homomorphisms (= K-linear ring homomorphisms) from A to $\overline{K}$. We make Γ act on $\Phi(A)$ by the formula $\gamma \cdot \phi = \gamma\phi : A \longrightarrow \overline{K}$, for all $\phi \in \mathrm{Alg}_K(A, \overline{K})$ and $\gamma \in \Gamma$. Recall that Γ is the automorphism group of the field $\overline{K}$ over K, so the composition $\gamma\phi$ makes sense.

It is easily seen that $\Phi(A_1 \times A_2)$ is the disjoint union of $\Phi(A_1)$ and $\Phi(A_2)$ (a homomorphism ϕ must map exactly one of the idempotents $(1,0)$ and $(0,1)$ to 1, and the other one to 0). If $A = L$ is a field finite over K, then the action of Γ on $\Phi(L)$ really happens through $G = \mathrm{Gal}(M/K) = \Gamma/\mathrm{Fix}(M)$, where M is any normal field extension of K which is again finite-dimensional. We also note that the cardinal of $\Phi(A)$ is the K-dimension of A, as is easily seen by reduction to the case that $A = L$ is a field.

Example 2.2.2. Let $K = \mathbb{Q}$ and $A = \mathbb{Q}(i)$. This is already a normal field extension. The set $\Phi(A)$ has two elements f_0 and f_1; one of them is the inclusion in $\overline{\mathbb{Q}}$, the other is complex conjugation. More generally, if $A = L = K(\alpha)$ where $p(x)$ is the minimal polynomial of α, then $\Phi(L)$ corresponds to the set $\{\alpha, \alpha_2, \ldots, \alpha_{deg(p)}\}$ of roots of $p(x)$ in the algebraic closure, just by looking at the image of α under f. This also shows that the cardinal of $\Phi(L)$ equals $[L : K]$; because of the compatibility with products, we have $|\Phi(A)| = \dim_K(A)$ in general.

Let us now define $\Psi : \mathcal{S}_\Gamma \longrightarrow \mathcal{A}_K$. Generally $\mathrm{Maps}(X, Y)$ denotes the set of mappings from X to Y (this was also written Y^X earlier). If both sets are Γ-sets, then we let $\mathrm{Maps}_\Gamma(X, Y) = \{f : X \longrightarrow Y | f(\gamma x) = \gamma f(x)\ \forall x \in X\ \forall \gamma \in \Gamma\}$. Define

$$\Psi(S) = \mathrm{Maps}_\Gamma(S, \overline{K}).$$

Via pointwise operations, $\Psi(S)$ becomes a commutative ring, and also a K-vector space; we will see its dimension is $|S|$. This K-algebra obviously has no nilpotents, so it is in $\mathcal{A}_K$.

The two operators are inverse to each other. We will show this and in the process gain a better understanding. Assume S is an orbit. Then $S \cong \Gamma/U$ with an open subgroup U. Let L be the fixed field of U. Then $[L : K] = [\Gamma : U]$. We claim $\Phi(L)$ identifies with S. Indeed via restriction, Γ surjects onto $\mathrm{Alg}(L, \bar{K}$, and $\gamma, \delta \in \Gamma$ become the same there iff their restrictions to L agree as maps; this in turn is equivalent with $\gamma^{-1}\delta$ being identity on L, that is, $\gamma^{-1}\delta \in U$, and this is finally the same as saying $\gamma U = \delta U$. On the other hand we claim that $\Psi(\Gamma/U)$ identifies with L. Indeed, for every $f \in \mathrm{Maps}_\Gamma(\Gamma/U, \bar{K}$, the element $x = f(e_\Gamma U$ bust be fixed under U, hence in L; on the other hand, f is determined by x, given that $f(\gamma U)$ must be $\gamma(x)$, and any $x \in L$ may take this role.

So we see that Φ and Ψ define inverse bijections between (finite) Γsets which are orbits on the one side, and K-algebras which are field on the other side. Now any Γ-set is the disjoint union of its orbits, and any algebra A is the product of fields. So

the claim about Φ and Ψ also hold for the larger domains where they are defined, given that our operators turn disjoint unions into cartesian products In passing we have also proved: $|\Phi(A)|$ equals the K-dimension of A.

We give some examples:

Example 2.2.3. Recall that for any open subgroup H (of finite index) in Γ, we saw that the fixed field L of H inside $\overline{K}$ corresponds to the Γ-set Γ/H.

Example 2.2.4. Let I be any finite set with trivial Γ-action (which means $\gamma i = i$ for all $\gamma \in \Gamma$, $i \in I$). What are then the Γ-invariant maps f from I to $\overline{K}$? All values of f must again be fixed under Γ, and the fixed field of Γ is the ground field K, so we get $\Psi(I) = \mathrm{Maps}(I, K) = K^I$ the direct product of copies of K, indexed by I. A special case of this is: The "trivial" algebra K corresponds to the one-point set. (Of course the operation on that set cannot be other than trivial.)

Example 2.2.5. Fix an integer $n > 1$, and choose a primitive n-th root ζ_n of unity in $\overline{K}$. We define the cyclotomic character $\omega : \Gamma \longrightarrow (\mathbb{Z}/n\mathbb{Z})^*$ by $\gamma(\zeta_n) = \zeta_n^{\omega(\gamma)}$. Using this we make $\mathbb{Z}/n\mathbb{Z}$ into a Γ-set, which will actually be considered as a Γ-group later on: we denote reduction mod n by an overbar and define

$$\gamma \cdot \overline{a} = \overline{\omega(\gamma)a}, \quad \overline{a} \in \mathbb{Z}/n\mathbb{Z}.$$

Denote by C_n a multiplicatively written cyclic group of order n, and pick a generator σ. Let $A = K[C_n]$ be the group ring; we have $A \cong K[x]/(x^n - 1)$ with σ mapping to $\overline{x}$.

We claim that $\Phi(A)$ is $\mathbb{Z}/n\mathbb{Z}$ with the cyclotomic Γ-action just defined. Indeed, the algebra homomorphisms from A to $\overline{K}$ are completely determined by the image of σ, and this can be any power of ζ_n. Thus, let $\phi_a : A \longrightarrow \overline{K}$ be the homomorphism that sends σ to ζ_n^a. If we apply γ, we get the homomorphism that sends σ to $\gamma(\zeta_n^a) = \zeta_n^{\omega(\gamma)a}$. Identifying ζ_n^a with $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ we get the claim.

Example 2.2.6. We have seen that Φ turns direct products of algebras into disjoint unions of sets. It is natural to ask: What corresponds to the direct product of sets on the algebra side? The answer is simple, nice and important: $\Phi(A \otimes B)$ can be naturally identified with $\Phi(A) \times \Phi(B)$, since every algebra homomorphism starting from $A \otimes B$ is uniquely characterized by what it does on $A = A \otimes 1$, and on $B = 1 \otimes B$.

At the end of this section, let us reconsider Hopf algebras in the light of this correspondence. We have not yet commented on the obvious fact that Φ and Ψ are not only defined on objects but also on maps (the technical details can safely be left to our readers); and both of the correspondence reverse the direction of the maps. Otherwise everything is preserved. Now a K-Hopf algebra H is just a K-algebra, with three extra algebra maps, which are (in order of decreasing complexity): the comultiplication $\Delta_H : H \longrightarrow H \otimes H$, the antipode $s_H : H \longrightarrow H$, and the augmentation $\varepsilon_H : H \longrightarrow K$. These maps must also obey certain axioms, coded as diagrams. The nice thing is now that we can mechanically translate all these things in the category of Γ-sets. Let $S = \Phi(H)$. Then:

- Δ_H gives $m_S : S \times S \longrightarrow S$;

- s_H gives $i_S : S \longrightarrow S$;
- $\varepsilon_H : H \longrightarrow K$ gives a map from the one-element set to S, that is: a distinguished element e_S of S.

From the nature of the diagrams it becomes clear without further effort that the Hopf axioms translate into saying that S is a group under m_S, with neutral element e_S and inverse map i_S. Furthermore, all maps on S etcetera are Γ-invariant. Let us define a Γ-group N to be a group N which is also a Γ-set, with the obvious compatibility condition that multiplication and formation of inverses commute with the Γ action and e_N is Γ-fixed. (This is actually a consequence.) We obtain:

Theorem 2.2.7. *There are inverse bijective correspondences Φ' and Ψ' between the category $\mathcal{H}_K$ of finite-dimensional commutative K-Hopf algebras on the one hand, and the category $\mathcal{G}_\Gamma$ of finite Γ-groups on the other. As before, the correspondences reverse all arrows; the product of Γ-groups corresponds to the tensor product of Hopf algebras.*

We give a few examples.

Example 2.2.8. Let us resume Example 2.2.4, assuming that the finite set I is a group (still with trivial Γ-action). Then $\Psi(I) = K^I$ becomes a Hopf algebra; let us look at the details, and we will recognize an old acquaintance . For $i \in I$ let $e_i \in K^I$ be the idempotent having 1 at position i and zero everywhere else; then $(e_i)_{i\in I}$ is a K-basis of K^I. From the definition of Ψ one can easily check the following:

$$\begin{aligned} \Delta e_i &= \sum_{j*k=i} e_j \otimes e_k; \\ s(e_i) &= e_{i^{-1}}; \\ \varepsilon(e_i) &= \delta_{i,1}. \quad \text{Kronecker's delta; 1 is the neutral element of } I \end{aligned}$$

Example 2.2.9. We go back to Example 2.2.5. We have the Hopf algebra $H = K[C_n]$ with $\Delta_H(\sigma) = \sigma \otimes \sigma$, $S_H(\sigma) = \sigma^{-1}$, and $\varepsilon_H(\sigma) = 1$. Recall that $S = \Phi(H) = \{\phi_0, \ldots, \phi_{n-1}\}$ where $\phi_i(\sigma) = \zeta_n^i$. We want to determine the group structure of S, which as a set was in canonical bijection with $\mathbb{Z}/n\mathbb{Z}$, so we expect that bijection to be also a group homomorphism. This is indeed the case: The product $\phi_i\phi_j$ in S is given by the composition

$$H \longrightarrow H \otimes H \longrightarrow \overline{K},$$

with the last map being $h \otimes h' \longmapsto \phi_i(h)\phi_j(h')$. Evaluated on σ, we get $\sigma \otimes \sigma$ and then $\phi_i(\sigma)\phi_j(\sigma)$, which is $\phi_{i+j}(\sigma)$. So indeed $\phi_i\phi_j = \phi_{i+j}$. This suffices to pin down the group structure. Recall that we already determined the Γ-action; one should spend a moment checking directly that the action is compatible with the group structure, as it has to be.

2.2 Translating Hopf-Galois structures and the Fix construction

We have a good understanding of algebras and Hopf algebras, via our correspondence. It will not be a surprise that the correspondence also applies to Hopf-Galois situations. Let us note two things: the resulting description is really simple, much

simpler than the original one (this is perhaps not surprising), and the coalgebra version (Hopf-Galois objects) is much more suitable for the translation than the algebra version (which is perhaps surprising at first).

Recall what it means that A is an H-Hopf-Galois object: we have a sort of diagonal $\beta\colon A \longrightarrow A \otimes H$ which is co-associative and co-unitary, and the induced map

$$\gamma : A \otimes A \longrightarrow A \otimes H, \quad a \otimes b \longmapsto (a \otimes 1) \cdot \beta(b)$$

is an isomorphism. (Equivalently, A is an H^*-Hopf-Galois extension, but this will be in the background for the moment.) We proceed to translate this into the language of Γ-sets. Let A correspond to the Γ-set S, and let H correspond to the Γ-group N.

Then β translates into a map $m = m_{S,N} : S \times N \longrightarrow S$. The axioms of coassociativity and co-unitarity are equivalent then to saying that m defines a (right) action of the group N on S, so S is a right N-set. (Recall that S is a left Γ-set.) We now ask ourselves what the bijectivity of γ means in terms of sets; the answer will be nice. As a preparation we need:

Definition 2.2.10. *Let Π be a group acting on a set X from the right. (Left actions can be treated similarly.) Then the action is transitive, if for any two $x, y \in Y$ there is $\pi \in \Pi$ with $x\pi = y$. The action is called simply transitive, when this π always exists, and is unique.*

Remark 2.2.11. The action is transitive iff X is an orbit, that is, isomorphic to $U\backslash\Omega$ for some subgroup U. The action is moreover simply transitive iff that subgroup is trivial. In other words: A set X with a simply transitive action of a group Ω is basically a copy of the group, only that in X we do not have a distinguished element, like the unit element in Ω.

Proposition 2.2.12. *With the above notation, the map γ is bijective if and only if the resulting action of N on S (on the right) is simply transitive.*

Proof. One mechanically translates γ into a map $q : S \times N \longrightarrow S \times S$, given by $q(s, \nu) = (s, s\nu)$. The bijectivity of q is then equivalent to the simple transitivity of the action of N on S. □

This situation is only possible if S and N have the same cardinality. We already know that these cardinalities are equal to the respective K-dimensions of K and H. So we recover the fact that a Hopf-Galois situation is only possible if the algebra and the Hopf algebra have the same dimension.

To complete the picture we revisit the Galois correspondence, that is, fixed and co-fixed subalgebras. As mentioned before, it is simpler to work with the comodule side. So assume that the algebra A is a J-Hopf-Galois object, and $g : J \longrightarrow J'$ is a surjective homomorphism of Hopf algebras. Let $S = \Phi(A)$, $N = \Phi(J)$, and $N' = \Phi(J')$. Then S has an action of N from the right which is simply transitive, and N' embeds as a subgroup of N (we consider this as an inclusion). Let $B = \mathrm{Fix}(g) \subset A$ be the co-fixed algebra; we want to understand $T = \Phi(B)$.

To do this we just have to translate the construction. As a set or vectorspace, B was defined as a difference kernel of two maps δ_0 and δ_1. That is, B is the largest subalgebra of A such that composing the inclusion $\iota : B \longrightarrow A$ with δ_0, and δ_1 respectively, gives the same map. Hence T is the finest surjective image of S such that composing $\Phi\delta_0$ (and $\Phi\delta_1$ respectively) with the surjection $S \longrightarrow T$ gives the same map. In other words, we are looking for the equivalence relation on S generated

by the postulate that $\Phi\delta_0(z)$ and $\Phi\delta_1(z)$ are equivalent, for all z in the domain of definition of the $\Phi\delta_i$, which is $S \times N'$. Now $\Phi\delta_0 : S \times N' \longrightarrow S$ is just the action of N on S, restricted to N'; and $\Phi\delta_1$ is the "no action" map, sending $(s, \nu) \longrightarrow s * 1_N = s$. Thus we are looking for the finest equivalence relation on S that makes s and $s * \nu$ equivalent, for all $\nu \in N'$.

This description is very concrete: T is just "S modulo N'", that is, the set of N'-orbits in S. This set T still has an action of N from the right. The fact that N acts simply transitively gives at once that all N'-orbits have $|N'|$ elements, so $|T| = |N|/|N'|$. We also see that T (or rather the equivalence relation defining it) allows to recover N'. We repeat these insights:

Theorem 2.2.13. *Let the notation be as above. Then we have an equality* $\dim_K(B) = \dim_K(J)/\dim_K(J')$. *Moreover the operator "co-fixed algebra" is injective, in the sense that surjections* $J \longrightarrow J'$ *and* $J \longrightarrow J''$ *that give rise to different subgroups* N', N'' *will also give rise to different co-fixed algebras.*

2.3 Base change

In this short section we take a different look at the (Hopf) algebras defined by Γ-sets, and Γ-groups, respectively. This view is often taken in the literature, and there it comes under the name "faithfully flat descent" or "Galois descent".

The correspondences defined in the preceding section depend on the base field K; in the present section it will be better to include this in the notation, writing Φ_K instead of Φ, and so on. Whenever L is a finite extension of K within $\overline{K}$, the algebraic closure of L is still $\overline{K}$, and $\Gamma_L = \mathrm{Aut}(\overline{K}/L)$ is an open subgroup of Γ_K. (Recall that if L is normal, then $G = \Gamma_K/\Gamma_L$ is the Galois group of L/K.)

We slightly rewrite the definition of Ψ_K. Remember that $\Psi_K(S)$ is the set of all Γ_K-equivariant maps $f : S \longrightarrow \overline{K}$. Actually $\mathrm{Maps}(S, \overline{K})$ is itself a Γ-set, by setting

$$(\gamma f)(s) = \gamma f(\gamma^{-1}s), \quad f : S \longrightarrow \overline{K}, s \in S.$$

When one checks that this does define a Γ_K-action, one will also see that one really needs to take inverses as written. But it is then clear that $\mathrm{Maps}_{\Gamma_K}(S, \overline{K})$ is then exactly the set of all $f \in \mathrm{Maps}(S, \overline{K})$ which are fixed under this new action.

For the next lemma (which is simple but fundamental) we need a harmless bit of notation: if X is any Γ_K-set, and L as above, then $X|L$ is the same set as X, but with restricted action: only Γ_L acts. It may seem unnecessary to indicate this, but the reader will see that it is useful for clarity.

Lemma 2.2.14. *With the above notations, we have for every commutative finite-dimensional K-algebra A the following:*

$$\Phi_L(L \otimes_K A) = \Phi_K(A)|L.$$

Proof. Again this will follow from the defining properties of the tensor product. Let us look at L-algebra homomorphisms $\phi' : L \otimes_K A \longrightarrow \overline{K}$. Then $\phi'(y \otimes a) = y \cdot \phi'(1 \otimes a)$ for all $y \in L$ and $a \in A$, so ϕ' is uniquely determined by its restriction ϕ to $1 \otimes A$, which we identify with A. This already identifies $\Phi_L(L \otimes A)$ with $\Phi_K(A)$ as sets. It is then obvious that the action of Γ_L is the same on both of these sets, now identified, which finishes the argument. □

The following will be formulated for commutative K-algebras, but everything holds also for comm. K-Hopf algebras with the appropriate changes. Consider a Γ-set S and the corresponding algebra A. There exists an open subgroup U of Γ such that H acts trivially on S, and we can even take U normal.

Let M be the fixed field of U; then $U = \Gamma_M$, and $G = \Gamma/U$ is the (finite) Galois group of M/K. By the lemma, $M \otimes A$ is the "trivial" algebra $M^S = \mathrm{Maps}(S, M)$, because the Γ_M-action on $\mathrm{Maps}(S, \overline{K})$ is just given by the action on $\overline{K}$, and the fixed field is M. The factor group G acts on $\mathrm{Maps}(S, M)$ in a way totally similar to the Γ_K-action on $\mathrm{Maps}(S, \overline{K})$: given $g \in G$ and $f : S \longrightarrow M$, we have $(gf)(s) = gf(g^{-1}s)$. Thus G acts by K-algebra automorphisms on $M \otimes A$, and the G-fixed subalgebra is A, for the following reason: Taking Γ_K-invariants at once is the same as first taking Γ_M-invariants and then taking $G = \Gamma_K/\Gamma_M$-invariants. Thus every comm. K-algebra A can be obtained from a "trivial" M-algebra by taking invariants under a suitable $\mathrm{Gal}(M/K)$-action, for a suitable finite Galois extension M/K. This M is also called a trivializing extension for A.

2.4 The so-called Greither-Pareigis correspondence

In this section, actions of Γ will be denoted by a dot $\cdot$ (or nothing), and an action of a Γ-group on a Γ-set will be denoted by $*$. The former is from the left, and the latter usually from the right.

Our classical example is $A = L$ a G-Galois extension of K, with the structure of K^G-Hopf-Galois object given by $\beta(x) = \sum_{g \in G} g(x) \otimes e_g$. The Γ-group N corresponding to K^G is the group G with trivial Γ-action; the Γ-set corresponding to L is $S = G = \Gamma/H$ where H is the group fixing L, with the obvious left Γ-action; and one checks that the action of G (as the group) on G (as the set) is again given by the group structure in G. This time the action is on the right.

Now let us look at a general situation: A is an H-Hopf-Galois object, with A corresponding to the Γ-set S and H corresponding to the Γ-group N. It is intentional that we don't use the letter G here, since we are not assuming that A is a G-Galois extension of K. By translation we get a simply transitive action $* : S \times N \longrightarrow S$. The map $N \longrightarrow \mathrm{Perm}(S)$ which sends ν to $\pi_\nu : S \ni s \longmapsto s * \nu$ is injective, and an anti-homomorphism of groups (if we use the usual composition as the group law in $\mathrm{Perm}(S)$. Thus, giving N and its action on S is the same as giving a simply transitive subgroup $\Pi = \{\pi_\nu : \nu \in N\}$ of $\mathrm{Perm}(S)$.

Let us denote the map $s \mapsto \gamma \cdot s$ (with $s \in S$ and $\gamma \in \Gamma$) by λ_γ. (Later this will indeed be a left multiplication.) The Γ-invariance of $*$ gives the following formula, for $\gamma \in \Gamma$, $\nu \in N$, and $s \in S$:

$$\lambda_\gamma(\pi_\nu(s)) = \pi_{\gamma \cdot \nu}(\lambda_\gamma(s)),$$

that is,

$$\pi_{\gamma \cdot \nu} = \lambda_\gamma \pi_\nu \lambda_\gamma^{-1},$$

or in terms of the group Π (we simply transfer the Γ-action from N to Π):

$$\gamma \cdot \phi = \lambda_\gamma \phi \lambda_\gamma^{-1}, \quad \forall \phi \in \Pi.$$

This shows that in our setting the Γ-action on Π (or N) can be determined from the other data, and moreover that Π as a subgroup of $\mathrm{Perm}(S)$ must be normalized by

all the λ_γ, with $\gamma \in \Gamma$. (If Ω is any group with any subgroup U, then $x \in \Omega$ is said to normalize U iff $xUx^{-1} = U$. The set $N_\Omega(U)$ of all x that normalize U is called the normalizer of U in Ω. It is the biggest subgroup of Ω which contains U as a normal subgroup.)

Now assume $A = L$ is a field. Then the Γ-set S becomes an orbit: it is Γ/Γ' with Γ' the open subgroup fixing L. (We have replaced U by Γ', to conform with the literature.) Then $\lambda_\gamma : \Gamma/\Gamma' \longrightarrow \Gamma/\Gamma'$ is indeed multiplication by γ on the left. We repeat what we have just seen:

Proposition 2.2.15. *Let $S = \Gamma/\Gamma'$ as above and let $\Pi \subset \mathrm{Perm}(S)$ be a simply transitive subgroup. Then the resulting action $* : S \times \Pi \longrightarrow S$ is Γ-equivariant if and only if the Γ-action on Π is given by the formula*

$$\gamma \cdot \pi = \lambda_\gamma \pi \lambda_\gamma^{-1}.$$

In particular Π must be normalized by all the left translations λ_γ.

Let us denote the subgroup of $\mathrm{Perm}(S)$ made up by all the λ_γ by Λ. We reformulate our findings as follows.

Theorem 2.2.16. *Let L/K be a field, finite over K, with fixed group $\Gamma' \subset \Gamma$. Then all instances of "L is a H-Hopf-Galois object" are given by simply transitive subgroups $\Pi \subset \mathrm{Perm}(\Gamma/\Gamma')$ such that Π is normalized by Λ. The Hopf algebra H is given by the group Π and the Γ-action via Λ (by conjugation).*

In the classical example where L/K is Galois with group G, the group Π is made up by all right translations ρ_γ as we have seen. Let us state this again, in different words: $G = \Gamma/\Gamma'$ (which is also S!!), the group G acts on the set G by right multiplication, so $\Pi = G$ acting by right multiplications on G. Here Π is not only normalized by Λ but actually centralized.

Let us revisit another example. Let $K = \mathbb{Q}$, p an odd prime, $a \in \mathbb{Q}$ not a p-th power. Let $\alpha = \sqrt[p]{a}$. Then $L = \mathbb{Q}(\alpha)$ has degree p; put $H = \mathbb{Q}[C$ where C is a cyclic group of order p. We have seen that $L/\mathbb{Q}$ is an H-Galois object. Let Γ' be the fixed group of L and let $\Gamma_0 \subset \Gamma'$ be the fixed group of the normal closure L' of L, which is given by $E = \mathbb{Q}(\alpha, \zeta_p)$. Finally write G for Γ/Γ_0; this is the Galois group of $L'/\mathbb{Q}$. It is instructive (if a bit involved) to determine G explicitly. Let $\sigma \in G$ be described by $\sigma(\alpha) = \zeta_p \alpha$ and $\sigma(\zeta_p) = \zeta_p$. On the other hand $\tau \in G$ is specified by saying that it fixes α and ζ_p to ζ_p^t where t is a chosen primitive root modulo p. Then G is the semidirect product of the cyclic group C of order p generated by σ,which is normal, and the cyclic group G' of order $p-1$ generated by τ. The action of the latter on the former is (only in different notation) the cyclotomic one, and G' is the image of Γ' in G, so $\Gamma/\Gamma' = G/G'$. We can identify G/G' with the set $S = \{0, 1, \ldots, p-1\}$, and the group Π (which is again cyclic of order p, with cyclotomic Γ-action) acts on this by cyclic shifts. Observe that $\tau \in G$ acts on S as multiplication by t. So this does not commute with the action of Π, but the group Π is normalized by τ which is "multiplication by t". In fact, the normalizer of the group Π (which is generated by the cyclic permutation $c : 0 \mapsto 1 \mapsto \cdots \mapsto p-1 \mapsto 0$) is exactly generated by c and τ, as we will prove later.

2.5 Explicit formulas

A variant of a previous example goes as follows (replace the odd prime p by the number 4): Take $a \in \mathbb{Q}$ squarefree, $a \neq \pm 1$. Take $L = \mathbb{Q}(x)$ with $x^4 = a$, and $J = \mathbb{Q}[C_4]$, where C_4 is cyclic of order 4 with chosen generator σ. Then one can show that L has degree 4, and $\beta : L \longrightarrow J \otimes L$, $x \longmapsto x \otimes \sigma$, makes L into a J-Galois object. For $S = \Phi(L)$ we get the set $\{0,1,2,3\}$ with a certain Γ-action, and $N = \mathbb{Z}/4\mathbb{Z}$ with the cyclotomic Γ-action.

On the module side, we have $H = J^* = \mathbb{Q}^{\mathbb{Z}/4\mathbb{Z}}$, which is the product of four copies of $\mathbb{Q}$, indexed by $0,1,2,3$. We have corresponding idempotents $e_0, \ldots, e_3$ (just one 1 and three zeros each), and the action of e_j on L is projection to the one-dimensional subspace $\mathbb{Q}x^j$. The same holds if we perform a base-change, that is we tensor everything with $E = \mathbb{Q}(i)$ over $\mathbb{Q}$; but then we should be careful and write $E \otimes L$ instead of $E(x)$ (even though one can show that these objects are equal, as $E(x)$ has degreee 8 over $\mathbb{Q}$). We define

$$\eta = e_0 + ie_1 - e_2 - ie_3 = (1, i, -1, -i) \in E \otimes H.$$

The following lemma is checked by calculation, using that we know the diagonal map on Hopf algebras of type K^N.

Lemma 2.2.17. *The element η is group-like, that is, $\Delta(\eta) = \eta \otimes \eta$. Note moreover that $\eta^4 = 1$.*

Now we define $c = \frac{1}{2}(\eta + \eta^3)$ and $s = \frac{1}{2i}(\eta - \eta^3)$. In quadruple notation we have $c = (1,0,-1,0)$ and $s = (0,1,0,-1)$. The action of c on L is certainly not an automorphism; but if restrict the action to the quadratic subfield

$$L_0 = \mathbb{Q} \oplus \mathbb{Q}x^2$$

, then c actually acts as the nontrivial automorphism of L_0 (you should convince yourself of this).

Lemma 2.2.18. 1. *$cs = 0$ and $c^2 + s^2 = 1$.*

2. *$\Delta c = c \otimes c - s \otimes s$ and $\Delta s = s \otimes c + c \otimes s$.*

Remark 2.2.19. These formula explain the choice of the letters; c and s are intended to be reminiscent of cosine and sine.

Proof. 1. The first formula is easy to show from the definitions, and actually obvious if we look at c and s written as quadruples.

2. We have $2\Delta\eta = \eta \otimes \eta + \eta^{-1} \otimes \eta^{-1}$. On the other hand, for $4(c \otimes c - s \otimes s)$ we get the eight-term sum $\eta \otimes \eta + \eta \otimes \eta^{-1} + \eta^{-1} \otimes \eta + \eta^{-1} \otimes \eta^{-1} + \eta \otimes \eta - \eta \otimes \eta^{-1} - \eta^{-1} \otimes \eta + \eta^{-1} \otimes \eta^{-1}$. After simplifying and comparing we obtain the first formula. The second formula is checked similarly.

□

We said that the element $c \in H$ does not act as a (field) automorphism. This is compatible with the fact that it is not group-like. However for $x, y \in L$ we have the

following formulas, which are reminiscent of the addition theorems for cosine and sine:

$$\begin{aligned} c(xy) &= c(x)c(y) - s(x)s(y); \\ s(xy) &= s(x)c(y) + c(x)s(y). \end{aligned}$$

It is open to debate whether these formulas are illuminating. It is certainly possible to perform similar computations in examples of larger dimension, but in our opinion the resulting formulas will not tell us much.

3 First applications of the main theorem

3.1 Almost classical extensions

This notion is inspired by the example $L = \mathbb{Q}(\sqrt[p]{a})$, whose normal closure is $L(\zeta_p)$. Here the group $G = \mathrm{Gal}(L(\zeta_p)/\mathbb{Q})$ can be split as a semidirect product, one factor of which is $\mathrm{Gal}(L(\zeta_p)/L)$. This is in fact a rather special situation. (Of course it arises in a trivial way if L/K is already a Galois extension itself.)

So assume that as always L/K is a finite-dimensional field extension with normal closure $\tilde{L}/K$. Let $G = \mathrm{Gal}(\tilde{L}/K)$, and let $G' < G$ be the subgroup $\mathrm{Gal}(\tilde{L}/L)$. So if Γ' is the subgroup of Γ fixing L, then the set of cosets Γ/Γ' identifies with G/G'. Assume moreover that there is a normal extension M/K inside $\tilde{L}$ such that

$$ML = \tilde{L},\ M \cap L = K.$$

The field M will be called a complement for L in $\tilde{L}$. Let $N < G$ be the group fixing M; this is a normal subgroup with $\mathrm{Gal}(M/K) = G/N$, and the intersection $N \cap G'$ is trivial. Better than that: G is the semidirect product $N \rtimes G'$. In the above example, the field M is $\mathbb{Q}(\zeta_p)$, and G is the semidirect product of two cyclic groups, the one of order $p-1$ acting on the one of order p, which is normal.

Let $P \subset \mathrm{Perm}(G/G')$ be the set (= subgroup) of all left translations λ_ν with $\nu \in N$. Recall $\Lambda = \{\lambda_\gamma : \gamma \in \Gamma\} \subset \mathrm{Perm}(G/G')$.

Proposition 2.3.1. *The group P acts simply transitively on G/G', and it is normalized by Λ. Therefore we obtain a Hopf-Galois object $L \longrightarrow L \otimes H$, where the Hopf algebra H belongs to the abstract group P with Γ-action via Λ.*

Proof. We first show that the action is transitive. It suffices that we can reach every class gU from $U = 1_G\dot{G}'$, by applying an element of P. Indeed we can decompose $g = \nu u$ with $\nu \in N$ and $u \in G'$, and then $\lambda_n u(1_G G') = \nu \cdot 1_G \cdot G') = \nu G' = gG'$. The uniqueness of ν is shown similarly; it follows from the fact that G' and N intersect trivially. Finally, P is normalized by Λ, because $\lambda_g \lambda_\nu \lambda_{g^{-1}} = \lambda_{g\nu g^{-1}}$, and $g\nu g^{-1} \in N$ since N is normal in G. □

Example 2.3.2. We revisit $L = \mathbb{Q}(\sqrt[p]{a})$ with hypotheses as before. Here we may take $M = \mathbb{Q}(\zeta_p)$, which is a normal (even abelian) extension of $\mathbb{Q}$ with degree $p-1$, so $M \cap L = \mathbb{Q}$, and we have already used that $ML = \widetilde{L} = L(\zeta_p)$ is the normal closure of $L/\mathbb{Q}$. The resulting Hopf-Galois structure coming from this "almost classical" setup is the same as the one explained before. Recall that the Γ-action on the cyclic group N of order p is the cyclotomic action.

Example 2.3.3. We take any non-normal cubic extension L/K. Then the Galois group G of $\widetilde{L}/K$ must be a copy of the symmetric group S_3, and $G' < G$ must be generated by a transposition. So we can take N to be the unique subgroup of order 3 in S_3; it is normal as is well known. Let us pin this down: "All cubic extensions are Hopf-Galois" (and even almost classically so).

Motivated by the last example, let us mention that there are extensions L/K which are not Hopf-Galois at all. Indeed there are many, but let us just discuss one class of examples. Let L/K be of degree 5 such that $\widetilde{L}/K$ has Galois group G isomorphic to the alternating group A_5. Then $S = G/G'$ is a 5-element set, on which G acts transitively, and in particular not trivially. So the resulting group homomorphism $\lambda : A_5 \cong G \longrightarrow \mathrm{Perm}(S)$ is a nontrivial homomorphism defined on a simple group, and therefore injective (the kernel is always a normal subgroup). That is, Λ is a copy of A_5 lying in $\mathrm{Perm}(S) \cong S_5$. So Λ is a subgroup of index 2 in S_5, hence normal; hence it contains all 5-cycles (look at the image in the group S_5/Λ of order 2). In fact Λ *is* A_5, but we don't need this. Now assume L/K is Hopf-Galois; this gives a simply transitive subgroup $N < \mathrm{Perm}(S)$ normalized by Λ. But then N has order 5, so it actually lies in Λ. On the other hand the simple group Λ does not normalize any nontrivial subgroup, contradiction.

3.2 The Byott translation

We keep the following setup: $\widetilde{L}$ is the normal closure of the finite extension L/K; the Galois group of $\widetilde{L}/K$ is G; and the subgroup belong to L is $G' < G$. Then G' contains no nontrivial normal subgroup of G, since otherwise $\widetilde{L}$ would not be the minimal normal over-field of L. One may always think of the example where $G = S_n$, and G' is the subgroup of all permutations that fix 1; then $S = G/G'$ identifies with $\{1, \ldots, n\}$; the dimensions are $[L : K] = n$ and $[\widetilde{L} : K] = n!$.

If one wants to exploit GP theory fully, it is hard to find the eligible subgroups $\Pi \subset S = \mathrm{Perm}(G/G')$. Byott's clever idea is to start with Π and look for G instead. Of course this takes some explanation: what is the suitable structure inside of which we may look for G? It is certainly not Π itself, that would be too simple. We begin with some abstract group theory, omitting the proofs of statements which will not really be used. In the following, let X be any group and $f : X \longrightarrow X$ be any bijective map. By $\mathrm{Aut}(X)$ we denote the set of all group automorphisms of X; this is again a group, under composition. For $x \in X$, the map $c_x : X \longrightarrow X$, $y \longmapsto xyx^{-1}$ is in $\mathrm{Aut}(X)$, and called conjugation by x. Recall that λ_v is left translation by an element $v \in X$.

Lemma 2.3.4. *The following are equivalent:*

(i) $f(xy^{-1}z) = f(x)f(y)^{-1}f(z)$, *for all* $x, y, z \in X$.

(ii) f *can be written* $f = \lambda_u \circ \phi$ *for some* $\phi \in \mathrm{Aut}(X)$, $u \in X$.

(iii) f *can be written* $f = \phi \circ \lambda_v$ *for some* $\phi \in \mathrm{Aut}(X)$, $v \in X$.

Proof. Most of the proof is easy and left to the reader. A few hints: Going from (ii) to (iii), ϕ stays the same, but v is not the same as u (what is it, exactly?) The implication (ii) to (i) is a calculation. Let us show how (i) $\implies$ (ii).

First step: The set of bijections f satisying (i) is closed under composition. (Fairly obvious.)

Second step: Every left multiplication λ_d satisfies (i). (Quick calculation.)

Final step: Assume f satisfies (i). Let $d = f(e_X)$ and put $g = \lambda_{d^{-1}} \circ f$. Then g again satisfies (i), and it has the extra property that it maps the neutral element e_X to itself. Putting $y = e_X$ in the equality (i), we get that g is a homomorphism of groups. □

Definition 2.3.5. *The subset of* $\mathrm{Perm}(X)$ *consisting of all* f *that satisfy one of the three conditions of the lemma is called the holomorph* $\mathrm{Hol}(X)$. *As already said, this subset is closed under composition, and in fact it is a subgroup.*

It is easily seen that the decomposition in item (ii) of the lemma is unique. If Λ_X denotes the subgroup of all λ_x, $x \in X$, then Λ_X is normalized by $\mathrm{Aut}(X)$ (see the exercises), and we get a representation of the holomorph as a semidirect product:

$$\mathrm{Hol}(X) = \Lambda_X \rtimes \mathrm{Aut}(X).$$

For later use we need a sharpening of this statement.

Proposition 2.3.6. $\mathrm{Hol}(X)$ *is the exact normalizer of* Λ_X *in* $\mathrm{Perm}(X)$.

Proof. We already know that $\mathrm{Aut}(X)$ normalizes Λ_X, and of course Λ_X normalizes itself. Putting these together we have that $\mathrm{Hol}(X)$ normalizes Λ_X. The point is to show the reverse inclusion. Assume f normalizes Λ_X. As in the proof of the lemma we write $f = \lambda g$, where λ is left multiplication by a suitable element, and g fixes $e = e_X$. Then g also normalizes Λ_X. Let us show that g is an automorphism. For any $x \in X$ there is $x' \in X$ such that $g\lambda_x g^{-1} = \lambda_{x'}$. Evaluating this in e we get $g(x) = x'$, so for all x we have the rule $g\lambda_x g^{-1} = \lambda_{g(x)}$. Now we take $x, y \in X$ and evaluate $w := g\lambda_{xy}g^{-1}$ two ways:

$$w = g\lambda_x g^{-1} g\lambda_y g^{-1} = \lambda_{g(x)}\lambda_{g(y)} = \lambda_{g(x)g(y)};$$

and

$$w = \lambda_{g(xy)}.$$

Evaluating w in e and using both these equalities shows that $g(x)g(y) = g(xy)$ as desired. □

A good example for this is given by the cyclic group $X = C$ of order p; we identify C with $\mathbb{Z}/p\mathbb{Z}$. The left multiplications (rather: additions!) Λ_C are then all the powers (rather: multiples) of the p-cycle $(0\,1\,\ldots\,p-1)$; this is again a copy of $\mathbb{Z}/p\mathbb{Z}$. The automorphisms of C are given as multiplications by integers prime to p; so $\mathrm{Aut}(C)$ is a copy of the unit group $\mathbb{Z}/p\mathbb{Z}^*$. The holomorph of C is a non-abelian group of order $p(p-1)$, and it is the exact normalizer of Λ_C.

Before reading on, please review the main result of GP theory. In the sequel we will write N instead of Π, to conform with the literature. The main idea of Byott is, very roughly: instead of having N permute G/G', we let a copy of G permute N. We set up some notation, and then we formulate and prove Byott's result. We keep the assumption that G is a finite group, G' a subgroup, and G' contains no nontrivial

normal subgroup of G. Moreover we still assume that N is a group of order $|G/G'|$. Define

$$\mathcal{N} = \{\alpha : N \longrightarrow \mathrm{Perm}(G/G') : \alpha(N) \text{ simply transitive}\};$$

and

$$\mathcal{G} = \{\beta : G \longrightarrow \mathrm{Perm}(N) : \beta(G') \text{ is the stabilizer of } e_N\}.$$

Theorem 2.3.7. *1. There is an explicit bijection between the sets $\mathcal{N}$ and $\mathcal{G}$ (described in the proof).*

2. If $\alpha \in \mathcal{N}$ corresponds to $\beta \in \mathcal{G}$ under that bijection, then $\alpha(N)$ is normalized by Λ_G if and only if $\beta(G)$ is contained in $\mathrm{Hol}(N)$.

Before we come to the proof, let us quickly explain why this is so useful: While $\mathrm{Perm}(G/G')$ is in general much larger that G/G', the holomorph $\mathrm{Hol}(N)$, while larger than N, is much smaller, comparatively seen.

Proof. As a small preparation, we observe that any bijection of sets $a : X \longrightarrow X'$ induces another bijection $Ca : \mathrm{Perm}(X) \longrightarrow \mathrm{Perm}(X')$, simply by putting $Ca(\pi) = a \circ \pi \circ a^{-1}$. (You might draw a little diagram for yourself, to visualize this.) – Moreover we will need that the left-multiplication map $\lambda : G \to \mathrm{Perm}(G)$ is injective. Indeed its kernel is normal in G, and contained in G', hence trivial, as said at the beginning of this subsection.

(a) We explain how α turns into β. Let α be given; by assumption it induces a bijection $a : N \longrightarrow G/G'$, via $a(\eta) = \alpha(\eta)(eG')$. Let $\lambda : G \longrightarrow \mathrm{Perm}(G/G')$ be our well-known left translation map, and define

$$\beta = Ca^{-1} \circ \lambda : \quad G \longrightarrow \mathrm{Perm}(G/G') \longrightarrow \mathrm{Perm}(N).$$

Then β is injective, as λ is injective (its kernel is normal in G and contained in G'), and Ca even bijective. The stabilizer of e_N under G (via β) is the stabilizer of eG' under G (via λ), and this is evidently G'. So the new map β is in the set $\mathcal{G}$.

(b) As a technical point, we claim and prove that $Ca^{-1} \circ \alpha : N \longrightarrow \mathrm{Perm}(N)$ is the same as the left translation map λ_N. This comes down to checking the commutativity of the following diagram for $\eta \in N$:

$$\begin{array}{ccc} G/G' & \xrightarrow{\alpha(\eta)} & G/G' \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle a} \\ N & \xrightarrow{\lambda_\eta} & N. \end{array}$$

We start with $\nu \in N$ in the southwest corner. For clarity, denote the class $e_G G'$ by $\bar{e}$. Going up and right, we get $\alpha(\nu)\bar{e}$, and then $\alpha(\eta)\alpha(\nu)\bar{e}$. Going first right and then up, we get $\eta\nu$ and then $\alpha(\eta\nu)\bar{e}$, and this is the same.

(c) Now we explain how β turns into α. Let $\beta : G \longrightarrow \mathrm{Perm}(N)$ be given with the indicated property. Then the orbit of e_N under G must be all of N, since G' is the stabilizer of e_N and the sets N and G/G' have the same cardinality.

This gives rise to a new bijection $b : G/G' \longrightarrow N$ via $gG' \longmapsto \beta(g)e_N$. As above, this induces the bijection $Cb : \mathrm{Perm}(G/G') \longrightarrow \mathrm{Perm}(N)$, and we put $\alpha = Cb^{-1} \circ \lambda_N : \quad N \longrightarrow \mathrm{Perm}(N) \longrightarrow \mathrm{Perm}(G/G')$. Again, we get immediately that the map α is injective. The image $\alpha(N)$ is simply transitive, because Λ_N is a simply transitive subgroup of $\mathrm{Perm}(N)$. Therefore $\alpha \in \mathcal{N}$ as required.

(d) The two constructions, from α to β, are mutually inverse: here we will be a bit shorter, and just say that if α leads to β, then the described bijections a and b are inverses of each other, and this is enough for checking that then β leads back to α.

(e) Now comes the final and central point: the equivalence of the additional property of α with that of β. – Assume first that $\alpha(N)$ is normalized by Λ_G, and β is constructed out of α as explained in step (1) above. Then $Ca^{-1}\alpha(N)$ is normalized by $Ca^{-1}\Lambda_G = \beta(G)$; by (2) we have $Ca^{-1}\alpha(N) = \lambda(N)$, and so $\lambda(N)$ is normalized by $\beta(G)$. By the proposition above (before the theorem), we conclude that $\beta(G) \subset \mathrm{Hol}(N)$. – Now assume that β is given, α is derived from it as explained in (c), and that $\beta(G) \subset \mathrm{Hol}(N)$. This says: $\lambda(N)$ is normalized by $\beta(G)$. Quite similarly as just before, this gives that $Cb^{-1}\lambda(N)$ is normalized by $Cb^{-1}\beta(G)$. The former is $\alpha(N)$ by construction; the latter is $\lambda(G)$, by the same technical argument as in (b) above. This shows the required extra condition on α.

□

Example 2.3.8. Let L/K be Galois in the classical sense. Then $\widetilde{L} = L$; $G = \mathrm{Gal}(L/K)$, and G' is trivial. This situation will be studied a lot later, but for now let us assume that G has order p (a prime number). We claim that there is only one Hopf-Galois structure for L/K. Indeed: in Byott's translation, the "other" group N must also be (cyclic) of order p. Therefore G must embed in $\mathrm{Hol}(N)$, which is known to us: it is the semidirect product of an order p group (which is normal) by a group or order $p-1$. Hence the p-Sylow subgroup of $\mathrm{Hol}(N)$ is normal, and unique, so there is only one choice for G. Thus there is only one choice on the other side (GP theory) as well, and it must be the classical one.

Example 2.3.9. Let $N = C_2 \times C_2$ (the non-cyclic group of order 4, which can also be seen as the two-dimensional $\mathbb{F}_2$-vectorspace). Then $\mathrm{Aut}(N) = \mathrm{GL}_2(\mathbb{F}_2)$ is non-abelian of order 6, and $\mathrm{Hol}(N)$ has order 24. As $\mathrm{Perm}(N)$ has only 24 elements as well, we have $\mathrm{Hol}(N) = \mathrm{Perm}(N)$. If we identify $\mathrm{Perm}(N)$ with S_4 (the details do not matter), any 4-cycle in $\mathrm{Hol}(N)$ generates a simply transitive subgroup G. That is: Every *cyclic* extension L/K of degree 4 admits a Hopf-Galois structure in which the involved group N is (of order 4 of course but) *non-cyclic.*

To finish this section we discuss a larger class of field extensions.

Theorem 2.3.10. *Assume $[L : K]$ is a prime number p, and let $G = \mathrm{Gal}(\widetilde{L}/K)$ as usual. Then L/K admits a Hopf-Galois structure if and only if G is solvable, and the latter happens exactly if G is a semidirect product $C \rtimes \Delta$, where C is of order p and Δ is a cyclic group of order dividing $p-1$.*

Proof. Assume that L/K has a Hopf-Galois structure. The group N such that G embeds into $\mathrm{Hol}(N)$ is also of order p, so $\mathrm{Hol}(N)$ is our old acquaintance $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}^*$, which is solvable. Hence G is also solvable, as a subgroup of a solvable group. Conversely, assume that G is solvable. By general Galois theory, G is a transitive subgroup of S_p, and (in particular) p divides $|G|$. By the Sylow theorem G contains a subgroup P of order p.

The following result is due to Galois; it is mentioned but not proved in the book of Childs [Chi00]. We will give a proof at the end of the section. Here is the statement.

Theorem 2.3.11 (Galois)**.** *A solvable subgroup G of S_p that contains an order p subgroup P is already contained in the normalizer of P, which can be identified with the holomorph of P.*

Now we assume the validity of the theorem: this shows our Galois group G lies between P and $\mathrm{Hol}(P)$, for a cyclic group P, and then we only have to take $N = P$ and appeal to the Byott translation. □

Proof. (of Theorem 2.3.11.) Assume the contrary, that is, P is not normal in G. As p^2 does not divide $|S_p|$, the subgroup P is a p-Sylow subgroup; if it is not normal, then G contains two (or more) subgroups of order p. The case $|G| = p$ (hence $G = P$) cannot occur. As G is solvable, G then contains a nontrivial subgroup H which is normal. Under the action of H, the set $\{1, \ldots, p\}$ splits up into disjoint orbits, which cannot all be trivial (singletons). On the other hand, G acts transitively on this orbit decomposition, so all H-orbits are of the same length. As p is prime, this is only possible if there is only one orbit, in other words: already H is transitive. Hence p divides $|H|$, and we can pick an order-p subgroup P' in H. Then P' is G-conjugate to all subgroups of order p in G, and there is more than one of them. As $P' \subset H$ and H is normal, all these conjugates lie already in H. We have shown: the statement "more than one subgroup of order p" is inherited from G down to H. But H is strictly smaller, and we may repeat the argument indefinitely. As our groups are finite, this is a contradiction. □

4 The Greither-Pareigis correspondence revisited

This section revolves around Theorem 2.2.16, the one commonly known as Greither-Pareigis theorem. In a few lines, if K is a field with algebraic closure $\overline{K}$ and $\Gamma = \mathrm{Gal}(\overline{K}/K)$, the theorem establishes that the equivalence from Section 2.1 between the categories $\mathcal{A}_K$ (finite-dimensional commutative K-algebras without nilpotent elements) and $\mathcal{S}_\Gamma$ (finite Γ-sets) defined by the maps Φ and Ψ restricts to a bijective correspondence between the Hopf-Galois structures on a separable extension of K with fixed subgroup Γ' and the simply transitive subgroups of $\mathrm{Perm}(\Gamma/\Gamma')$ normalized by left translations of Γ/Γ'. Most of the importance in this result lies in the fact that it ties the determination of Hopf-Galois structures on separable extensions with group theory. In this section, we will reformulate the theorem in a way that is more convenient for many applications, and we shall see the explicit form of the correspondence.

4.1 An alternative glance to the main theorem

We start by rewriting Theorem 2.2.16 in a convenient way to work with.

Let L/K be a separable field extension with algebraic closure $\overline{K}$. Call $\Gamma = \mathrm{Gal}(\overline{K}/K)$ and $\Gamma' = \mathrm{Gal}(\overline{L}/L)$. As already mentioned, Greither-Pareigis theorem establishes an one-to-one correspondence between Hopf-Galois structures on L/K and the subgroups of $\mathrm{Perm}(\Gamma/\Gamma')$ that are simply transitive and normalized by the set $\overline{\Lambda}$ of left translations by elements $\gamma \in \Gamma$.

First, simply transitive subgroups of $\mathrm{Perm}(\Gamma/\Gamma')$ are, by definition, those whose group action on Γ/Γ' is simply transitive. From now on, we shall refer to such subgroups as **regular**. For later use, we see some characterizations of this concept.

Proposition 2.4.1. *Let X be a finite set and let N be a subgroup of $\mathrm{Perm}(X)$. Consider the group action of N on X defined by evaluation. If two of the following three conditions are satisfied, so is the other one.*

1. $|N| = |X|$.
2. *N acts transitively on X.*
3. *Given $x \in X$, $\mathrm{Stab}_N(x) = \{\eta \in N \mid \eta(x) = x\} = \{1_N\}$.*

Proof. Fix $x \in X$. By the orbit-stabilizer theorem, we have $|N| = |\mathrm{Orb}(x)|\,|\mathrm{Stab}_N(x)|$. Now, let us note that 2 is equivalent to $|\mathrm{Orb}(x)| = |X|$ and 3 is equivalent to $|\mathrm{Stab}_N(x)| = 1$. Then the statement follows immediately. □

If X is a finite set and N is a subgroup of $\mathrm{Perm}(X)$, for each $x \in X$ we consider the map $\varphi_x \colon N \longrightarrow X$ defined by $\varphi_x(\eta) = \eta \cdot x$.

Proposition 2.4.2. *Let X be a finite set and let N be a subgroup of $\mathrm{Perm}(X)$. The following conditions are equivalent.*

1. *N is a regular subgroup of $\mathrm{Perm}(X)$.*
2. *Two of the conditions from Proposition 2.4.1 are satisfied.*
3. *The conditions from Proposition 2.4.1 are satisfied.*
4. *There is some $x \in X$ such that φ_x is bijective.*
5. *For every $x \in X$, φ_x is bijective.*

Proof. The equivalence between 2 and 3 has been already shown in Proposition 2.4.1.

Suppose that 1 holds, so that N acts simply transitively on X. In particular, the action is transitive. Let us fix $x \in X$. Then, for each $y \in X$ there is a unique $\eta_y \in N$ such that $\eta_y(x) = y$. By the uniqueness, the η_y define $|X|$ different elements in N, and they are all the elements of N (given $\eta \in N$, $\eta = \eta_{\eta(x)}$), so $|N| = |X|$. Hence 2 is satisfied. Conversely, assume that 3 holds. Let $x, y \in X$. Since N acts transitively on X, there is $\eta \in N$ such that $\eta(x) = y$. Suppose that $\mu \in N$ is such that $\mu(x) = y$. Then $\eta(x) = \mu(x)$, whence $\eta^{-1}\mu(x) = x$, that is, $\eta^{-1}\mu \in \mathrm{Stab}_N(x) = \{1_N\}$. Hence $\eta = \mu$, proving that the action is simply transitive.

Let us prove that 1 and 5 are equivalent. Given $x \in X$, we have that the map φ_x is bijective if and only if there is a unique $\eta \in N$ such that $\eta \cdot x = y$, whence the

claim follows. On the other hand, it is trivial that 5 implies 4. Finally, assume that 4 is satisfied, so that for some $x \in X$, φ_x is bijective. Then for each $y \in X$ there is a unique $\eta \in N$ such that $\eta \cdot x = y$, so N acts simply transitively on X and 1 holds. □

On the other hand, in Section 3, we have used an alternative quotient set G/G' of Galois groups, that comes from choosing the normal closure of our separable extension L/K, instead of its algebraic closure. This is valid because the left cosets of Γ/Γ' and G/G' can be identified. In the following we offer a complete proof for the validity of this step.

Proposition 2.4.3. *Let L/K be a finite and separable extension of fields and let E/K be a Galois extension with $L \subset E$. Call $G_E = \mathrm{Gal}(E/K)$ and $G'_E = \mathrm{Gal}(E/L)$. The Hopf-Galois structures on L/K are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G_E/G'_E)$ normalized by the set Λ of left translations by elements $g \in G$.*

Proof. We know by Theorem 2.2.16 that the Hopf-Galois structures on L/K are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(\Gamma/\Gamma')$ normalized by the set $\overline{\Lambda}$ of left translations by elements $\gamma \in \Gamma$. We shall prove that the latter are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G_E/G'_E)$ normalized by Λ, whence the statement will follow.

Since E/K is Galois, by Theorem 1.1.58, $G(E) := \mathrm{Gal}(\overline{L}/E)$ is a normal subgroup of Γ and the restriction maps $\Gamma \longrightarrow G_E$, $\Gamma' \longrightarrow G'_E$ induce group isomorphisms

$$\Gamma/G(E) \cong G_E, \quad \Gamma'/G(E) \cong G'_E.$$

Then, the map $\varphi\colon \Gamma/\Gamma' \longrightarrow G_E/G'_E$ defined by $\varphi(\gamma\Gamma') = \gamma \mid_E G'_E$ is bijective. At the same time, such a map induces a group isomorphism $\Phi\colon \mathrm{Perm}(\Gamma/\Gamma') \longrightarrow \mathrm{Perm}(G_E/G'_E)$ defined as $\Phi(\eta)(\varphi(\gamma\Gamma')) = \varphi(\eta(\gamma\Gamma'))$. It is enough to check that a subgroup of $\mathrm{Perm}(\Gamma/\Gamma')$ is regular and normalized by $\overline{\Lambda}$ if and only if it is mapped by Φ to a regular subgroup of $\mathrm{Perm}(G_E/G'_E)$ normalized by Λ.

Let N be a regular subgroup of $\mathrm{Perm}(\Gamma/\Gamma')$ and let us prove that $\Phi(N)$ is regular. Let $a, b \in G_E/G'_E$ and write $x = \varphi^{-1}(a)$ and $y = \varphi^{-1}(b)$. Since N is regular and $x, y \in \Gamma/\Gamma'$, there is a unique $\eta \in N$ such that $\eta(x) = y$. Now, $\Phi(\eta)(a) = \Phi(\eta)(\varphi(x)) = \varphi(\eta(x)) = \varphi(y) = b$. The uniqueness of $\Phi(\eta)$ follows from the bijectivity of Φ. Hence $\Phi(N)$ is regular. The converse is proved in the same way.

Let N be a subgroup of $\mathrm{Perm}(\Gamma/\Gamma')$ normalized by $\overline{\Lambda}$. Given $\gamma, \mu \in \Gamma$, we have

$$\lambda_{\gamma|_E} \circ \varphi(\mu\Gamma') = \lambda_{\gamma|_E}(\mu \mid_E G'_E) = (\gamma\mu) \mid_E G'_E = \varphi(\gamma\mu\Gamma') = \varphi \circ \lambda_\gamma(\mu\Gamma').$$

Since μ is arbitrary, we obtain that $\lambda_{\gamma|_E} \circ \varphi = \varphi \circ \lambda_\gamma$. Let us check that $\lambda_{\gamma|_E} \circ \Phi(N) \circ \lambda_{\gamma|_E}^{-1} \subseteq \Phi(N)$. Let $\eta \in N$. For an arbitrary $g \in G_E$, let $\mu \in \Gamma$ be such that $g = \mu \mid_E$. Then

$$\begin{aligned}
\lambda_{\gamma|_E} \circ \Phi(\eta) \circ \lambda_{\gamma|_E}^{-1}(gG'_E) &= \lambda_{\gamma|_E} \circ \Phi(\eta)((\gamma^{-1}\mu) \mid_E G'_E) \\
&= \lambda_{\gamma|_E} \circ \Phi(\eta)(\varphi(\gamma^{-1}\mu\Gamma')) \\
&= \lambda_{\gamma|_E} \circ \varphi \circ \eta(\gamma^{-1}\mu\Gamma') \\
&= \varphi \circ \lambda_\gamma \circ \eta \circ \lambda_{\gamma^{-1}}(\mu\Gamma') \\
&= \Phi(\lambda_\gamma \circ \eta \circ \lambda_{\gamma^{-1}})(\varphi(\mu\Gamma')) \\
&= \Phi(\lambda_\gamma \circ \eta \circ \lambda_{\gamma^{-1}})(gG'_E).
\end{aligned}$$

Since g is arbitrary, $\lambda_{\gamma|_E} \circ \Phi(\gamma) \circ \lambda_{\gamma|_E}^{-1} = \Phi(\lambda_\gamma \circ \eta \circ \lambda_{\gamma^{-1}})$. Now, since N is normalized by left translations by hypothesis, we have $\lambda_\gamma \circ \eta \circ \lambda_{\gamma^{-1}} \in N$, so $\lambda_{\gamma|_E} \circ \Phi(\gamma) \circ \lambda_{\gamma|_E}^{-1} \in \Phi(N)$, as we wanted. We conclude that $\Phi(N)$ is normalized by Λ. The converse is proved likewise. □

Proposition 2.4.3 means that, in order to characterize Hopf-Galois structures on a separable extension L/K in terms of permutation subgroups, instead of choosing an algebraic closure to construct the Galois groups Γ and Γ', we can just choose any finite and Galois extension of E containing L, and choose the corresponding Galois groups G_E and G'_E.

The remaining ingredient concerning Theorem 2.2.16 is left translations of Γ/Γ'. We have proved in Proposition 2.4.3 that, for any Galois extension E of K containing L, we can consider instead the set of left translations $\lambda_g \colon hG'_E \mapsto ghG'_E$ of G_E/G'_E, where G_E and G'_E are in the statement of that result. We can regard this as the image of a map.

Definition 2.4.4. *Let L/K be a finite and separable extension, let E/K be a Galois extension with $L \subset E$ and acquire the above notation. The **left translation map** of L/K associated to E is the map*

$$\begin{array}{rccc} \lambda_E \colon & G_E & \longrightarrow & G_E/G'_E \\ & g & \longrightarrow & hG'_E \mapsto ghG'_E \end{array}$$

The left translation map is not in general injective, and its kernel can be characterized in terms of group theory.

Definition 2.4.5. *Let G be a group and let G' be a subgroup of G. The **core** of G' inside G is defined as*

$$\mathrm{Core}_G(G') = \bigcap_{g \in G} gG'g^{-1}.$$

In other words, it is the greatest normal subgroup of G contained in G'.

Proposition 2.4.6. *Let L/K be a finite and separable extension, and let E/K be a Galois extension with $L \subseteq E$. Call $G_E = \mathrm{Gal}(E/K)$, $G'_E = \mathrm{Gal}(E/L)$, and let $\lambda_E \colon G_E \longrightarrow G_E/G'_E$ be the left translation map of L/K associated to E. Then*

$$\mathrm{Ker}(\lambda_E) = \mathrm{Core}_{G_E}(G'_E).$$

Proof. Let $h \in G_E$. We have that

$$\begin{aligned} h \in \mathrm{Ker}(\lambda_E) &\iff \lambda_E(h) = \mathrm{Id}_{G_E/G'_E} \\ &\iff hgG'_E = gG'_E \text{ for all } g \in G_E \\ &\iff g^{-1}hgG'_E = G'_E \text{ for all } g \in G_E \\ &\iff h \in gG'_Eg^{-1} \text{ for all } g \in G_E \\ &\iff h \in \mathrm{Core}_{G_E}(G'_E) \end{aligned}$$

□

Let L/K be a finite and separable field extension. Note that the smallest field E such that $L \subset E$ is by definition the normal closure $\widetilde{L}$ of L/K. This will be our preferred choice when we make use of Greither-Pareigis theorem. Call $G = \mathrm{Gal}(\widetilde{L}/K)$ and $G' = \mathrm{Gal}(\widetilde{L}/L)$. In short, we will say that L/K is (G, G')-*separable* or G-*separable*. In this case, the left translation map $\lambda\colon G \longrightarrow G/G'$ of L/K associated to $\widetilde{L}$ is simply called the left translation map of L/K. If no more quotient groups arise, we will normally write left cosets of G/G' as $\overline{g}$ for a representative $g \in G$. Thus, for $g, h \in G$, $\lambda(g)(\overline{h}) = \lambda_g(\overline{h}) = \overline{gh}$.

Corollary 2.4.7. *The left translation map λ of a (G, G')-separable extension L/K is injective.*

Proof. We know from Proposition 2.4.6 that $\mathrm{Ker}(\lambda) = \mathrm{Core}_G(G')$, which is by definition the greatest normal subgroup of G contained in G'. By definition of normal closure, $\widetilde{L}$ is the smallest Galois field extension of K containing L. In other words, there are no Galois extensions of K containing L and properly contained in $\widetilde{L}$. Applying the Galois correspondence, we get that there are no non-trivial normal subgroups of G contained in G'. That is, $\mathrm{Core}_G(G') = \{\overline{1_G}\}$, proving the statement. □

Let us focus on the normality condition for a permutation subgroup at the Greither-Pareigis correspondence. Let L/K be a (G, G')-separable extension and let $\lambda\colon G \longrightarrow \mathrm{Perm}(G/G')$ be its left translation map. Since λ is injective, G is isomorphic with its image $\lambda(G)$, which is a subgroup of $\mathrm{Perm}(G/G')$. We have an action of G on $\mathrm{Perm}(G/G')$ by letting $\lambda(G)$ act by conjugation:

$$g \cdot \eta := \lambda(g)\eta\lambda(g^{-1}), \quad \eta \in \mathrm{Perm}(G/G').$$

The condition that a subgroup N of $\mathrm{Perm}(G/G')$ is normalized by the left translations is just that this action restricts to N.

Definition 2.4.8. *Let N be a subgroup of $\mathrm{Perm}(G/G')$. We say that N is G-**stable**, or that N is normalized by $\lambda(G)$, if for every $g \in G$ and $\eta \in N$,*

$$\lambda(g)\eta\lambda(g^{-1}) \in N,$$

that is, $\lambda(G)$ acts on N by conjugation.

Under this terminology, we can restate Theorem 2.2.16 as follows.

Theorem 2.4.9. *Let L/K be a (G, G')-separable extension. Then, there is a bijective correspondence between:*

1. *The Hopf-Galois structures on L/K.*
2. *The regular and G-stable subgroups of $\mathrm{Perm}(G/G')$.*

We also give a term for an concept that has already appeared; namely, the isomorphism class of a permutation subgroup corresponding to a Hopf-Galois structure on a separable extension.

Definition 2.4.10. *The **type** of a Hopf-Galois structure H on a (G, G')-separable extension is defined as the isomorphism class of the subgroup N of $\mathrm{Perm}(G/G')$ corresponding to H under the Greither-Pareigis correspondence. We denote it by $[N]$.*

We can classify Hopf-Galois structures on a separable extension according to their type. We saw that Byott's translation allows us to count Hopf-Galois structures of a given type on a separable extension.

4.2 The explicit form of the correspondence

Let L/K be a (G, G')-separable extension with normal closure $\widetilde{L}$. In this part we describe the definition of the bijective (and inverse-to-each-other) maps involved in the Greither-Pareigis correspondence. The following establishes a first relation between a Hopf-Galois structure H on L/K and its corresponding permutation subgroup N.

Proposition 2.4.11 ([GP87], Proposition 1.3)**.** *Let L/K be a (G, G')-separable extension with normal closure $\widetilde{L}$. Let H be a Hopf-Galois structure on L/K and let N be its corresponding regular and G-stable subgroup of* $\mathrm{Perm}(G/G')$*. Then $\widetilde{L} \otimes_K H \cong \widetilde{L}[N]$ as $\widetilde{L}$-Hopf algebras.*

First, we see how to recover H from N. To do so, we need some notions from Galois descent theory. First, it is easy to check that the K-Hopf algebras together with the homomorphisms of K-Hopf algebras form a category. The same is true for $\widetilde{L}$-Hopf algebras, but we shall consider a smaller category inside.

Let M be an $\widetilde{L}$-Hopf algebra. An $\widetilde{L}$-semilinear action of G on M is defined as a map $*\colon \widetilde{L}[G] \otimes_{\widetilde{L}} M \longrightarrow M$ such that for every $g \in G$, the map $g * -\colon M \longrightarrow M$ is $\widetilde{L}$-semilinear, that is, there is some field automorphism $\sigma_g \in \mathrm{Aut}(L)$ such that

$$g * (\lambda m) = \sigma_g(\lambda) g * m, \quad \lambda \in \widetilde{L},\ m \in M.$$

If there are $\widetilde{L}$-semilinear actions of G on $\widetilde{L}$-Hopf algebras M, M' respectively, an $\widetilde{L}$-linear map $f\colon M \longrightarrow M'$ is said to be G-equivariant if

$$g * f(m) = f(g * m), \quad g \in G,\ m \in M.$$

Definition 2.4.12. *Let M be an $\widetilde{L}$-Hopf algebra endowed with an $\widetilde{L}$-semilinear action from G. Consider the induced $\widetilde{L}$-semilinear action of G on $M \otimes_{\widetilde{L}} M$ as*

$$g * (m \otimes m') := (g * m) \otimes (g * m'), \quad g \in G,\ m, m' \in M.$$

We say that M is G-compatible if all the Hopf algebra operations of M are G-equivariant maps.

The G-compatible $\widetilde{L}$-Hopf algebras form a category where the morphisms are the G-equivariant $\widetilde{L}$-Hopf algebra homomorphisms.

Definition 2.4.13. *Let M be a G-compatible $\widetilde{L}$-Hopf algebra and write $*$ for the action of G on M. The sub-Hopf algebra of M fixed by G is*

$$M^G := \{m \in M \mid g * m = m\}.$$

The main result for our purposes is the following:

Theorem 2.4.14. *Let L/K be a separable extension with normal closure $\widetilde{L}$ and let $G = \mathrm{Gal}(\widetilde{L}/K)$.*

1. *If H is a K-Hopf algebra, then $\widetilde{L} \otimes_K H$ is a G-compatible $\widetilde{L}$-Hopf algebra.*
2. *If M is a G-compatible $\widetilde{L}$-Hopf algebra, then M^G is a K-Hopf algebra.*

Moreover, these assignments define an equivalence of categories between the category of K-Hopf algebras and the category of G-compatible $\widetilde{L}$-Hopf algebras.

This is explained at [Chi00, Paragraph before (2.13)].

As a consequence, for a G-compatible $\widetilde{L}$-Hopf algebra M, $\widetilde{L} \otimes M^G \cong M$ as G-compatible $\widetilde{L}$-Hopf algebras. Likewise, for a K-Hopf algebra H, $(\widetilde{L} \otimes_K H)^G \cong H$ as K-Hopf algebras.

Let N be a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$. Let λ be the left translation map of L/K. That N is G-stable means that N is normalized by $\lambda(G)$, or equivalently, the conjugation action of G on $\mathrm{Perm}(G/G')$ leaves N invariant. We can easily extend this action to an $\widetilde{L}$-semilinear action of G on $\widetilde{L}[N]$ by letting G act on $\widetilde{L}$ by means of the usual Galois action and on N by the action above. Explicitly,

$$g * \Big(\sum_{i=1}^{n} h_i \eta_i \Big) = \sum_{i=1}^{n} g(h_i) \lambda(g) \eta_i \lambda(g^{-1}), \tag{2.1}$$

where $g \in G$, $n \in \mathbb{Z}_{>0}$ and, for each $1 \leq i \leq n$, $a_i \in \widetilde{L}$ and $\eta_i \in N$. This is indeed semilinear: if $g \in G$, $\lambda \in \widetilde{L}$ and $h = \sum_{i=1}^{n} h_i \eta_i \in \widetilde{L}[N]$, then

$$g * (\lambda h) = g * \Big(\sum_{i=1}^{n} \lambda h_i \eta_i \Big) = \sum_{i=1}^{n} g(\lambda) g(h_i) \lambda(g) \eta_i \lambda(g^{-1}) = g(\lambda) g * h.$$

Proposition 2.4.15. *Let L/K be a (G, G')-separable extension with normal closure $\widetilde{L}$. If N is a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$, the $\widetilde{L}$-group algebra $\widetilde{L}[N]$ is a G-compatible $\widetilde{L}$-Hopf algebra with respect to the action $*$ of G on $\widetilde{L}[N]$ defined at (2.1).*

Proof. We need to check that the Hopf algebra operations of $\widetilde{L}[N]$ are G-equivariant.

- Multiplication: Given $h = \sum_{i=1}^{n} h_i \eta_i, h' = \sum_{j=1}^{n} h'_j \eta_j \in \widetilde{L}[N]$ and $g \in G$,

$$\begin{aligned} g * m_{\widetilde{L}[N]}(h \otimes h') &= g * \sum_{i,j=1}^{n} h_i h'_j \eta_i \eta_j \\ &= \sum_{i,j=1}^{n} g(h_i h'_j) \lambda(g) \eta_i \eta_j \lambda(g^{-1}) \\ &= \sum_{i,j=1}^{n} g(h_i) g(h'_j) \lambda(g) \eta_i \lambda(g^{-1}) \lambda(g) \eta_j \lambda(g^{-1}) \\ &= \Big(\sum_{i=1}^{n} g(h_i) \lambda(g) \eta_i \lambda(g^{-1}) \Big) \Big(\sum_{j=1}^{n} g(h'_j) \lambda(g) \eta_j \lambda(g^{-1}) \Big) \\ &= (g * h)(g * h') \\ &= m_{\widetilde{L}[N]}((g * h) \otimes (g * h')) \\ &= m_{\widetilde{L}[N]}(g * (h \otimes h')) \end{aligned} \tag{2.2}$$

- Unit: Given $r \in K$ and $g \in G$,

$$g * u_{K[G]}(r) = g * (r 1_G) = r 1_G = u_{K[G]}(g * r).$$

- Comultiplication: Let $h = \sum_{i=1}^{n} h_i\eta_i \in \widetilde{L}[N]$ and $g \in G$. Then,

$$\begin{aligned} g * \Delta_{\widetilde{L}[N]}(h) &= g * \Big(\sum_{i=1}^{n} h_i\eta_i \otimes \eta_i \Big) \\ &= \sum_{i=1}^{n} g(h_i)\lambda(g)\eta_i\lambda(g^{-1}) \otimes \lambda(g)\eta_i\lambda(g^{-1}) \\ &= \Delta_{\widetilde{L}[N]}\Big(\sum_{i=1}^{n} g(h_i)\lambda(g)\eta_i\lambda(g^{-1}) \Big) \\ &= \Delta_{\widetilde{L}[N]}(g * h). \end{aligned}$$

- Counit: For $h = \sum_{i=1}^{n} h_i\eta_i \in \widetilde{L}[N]$ and $g \in G$, we have

$$g * \varepsilon_{\widetilde{L}[N]}(h) = g * \Big(\sum_{i=1}^{n} h_i \Big) = \sum_{i=1}^{n} g(h_i) = \varepsilon_{\widetilde{L}[N]}(g * h)$$

- Coinverse: Again, given $h = \sum_{i=1}^{n} h_i\eta_i \in \widetilde{L}[N]$ and $g \in G$, we have

$$\begin{aligned} g * S_{\widetilde{L}[N]}(h) &= g * \sum_{i=1}^{n} h_i\eta_i^{-1} \\ &= \sum_{i=1}^{n} g(h_i)\lambda(g)\eta_i^{-1}\lambda(g^{-1}) \\ &= \sum_{i=1}^{n} g(h_i)(\lambda(g)\eta_i\lambda(g^{-1}))^{-1} \\ &= S_{\widetilde{L}[N]}(g * h). \end{aligned}$$

□

Taking into account Proposition 2.4.11, we obtain an explicit description for the underlying Hopf algebra. The action is also obtained by descent. We summarize what we get at the following.

Proposition 2.4.16. *Let L/K be a (G, G')-separable extension and let N be a regular and G-stable subgroup of* $\mathrm{Perm}(G/G')$. *Let H be the Hopf-Galois structure on L/K that corresponds to N under the Greither-Pareigis correspondence.*

1. *The underlying Hopf algebra of H is*

$$\widetilde{L}[N]^G = \{h \in \widetilde{L}[N] \mid g * h = h \text{ for all } g \in G\}.$$

2. *The action of H on L is given as follows: For $h = \sum_{i=1}^{n} h_i\eta_i \in H$ and $\alpha \in L$,*

$$h \cdot \alpha = \sum_{i=1}^{n} h_i\eta_i^{-1}(\overline{1})(\alpha), \tag{2.3}$$

where for each $1 \leq i \leq n$, $\eta_i^{-1}(\overline{1})(\alpha)$ is the image of α by a representative g of the left coset $\eta_i^{-1}(\overline{1}) \in G/G'$.

.

Let us check that the expression 2.3 is well defined. Take two representatives $g, k \in G$ of the left coset $\eta_i^{-1}(\overline{1})$ and an element $\alpha \in L$. Since g and k belong to the same left coset, $g^{-1}k \in G' = \mathrm{Gal}(\widetilde{L}/L)$, so $\alpha = g^{-1}k(\alpha)$, that is, $g(\alpha) = k(\alpha)$.

The correspondence in the converse direction follows easily from Proposition 2.4.11. Indeed, if H is a Hopf-Galois structure on a separable extension L/K with normal closure $\widetilde{L}$ and N is its corresponding subgroup, we have that $\widetilde{L} \otimes_K H \cong \widetilde{L}[N]$ as $\widetilde{L}$-Hopf algebras. By Corollary 1.2.19, N can be regarded as the group of grouplike elements of $\widetilde{L} \otimes_K H$.

4.3 The Greither-Pareigis theorem for Galois extensions

In this section we deepen in the specification of Greither-Pareigis theorem for Galois extensions from Section 2.4 so as to visualize the group-theoretical description of all their Hopf-Galois structures.

Let L/K be a Galois extension with group G. We know that $K[G]$ together with its classical action on L is a Hopf-Galois structure on L/K. We will often refer to this as the **classical Galois structure**.

By definition, the normal closure of L/K is $\widetilde{L} = L$. Thus, in this case, the groups G and G' appearing at the statement of Theorem 2.4.9 are $G = \mathrm{Gal}(L/K)$ and $G' = \{\mathrm{Id}_G\}$. In other words, L/K is $(G, \{\mathrm{Id}_G\})$-separable. Thus, Theorem 2.4.9 becomes:

Theorem 2.4.17. *Let L/K be a Galois extension with group G. There is a bijective correspondence between:*

- *The regular and G-stable subgroups of* $\mathrm{Perm}(G)$.
- *The Hopf-Galois structures on L/K.*

Let us specify what G-stable means in the Galois case. Following Definition 2.4.8, a subgroup $N \leq \mathrm{Perm}(G)$ is G-stable if the action of G on $\mathrm{Perm}(G)$ leaves N invariant. Such an action is defined by conjugation with the image of G by the left translation of L/K from Definition 2.4.4. Since $G' = \{1_G\}$, the left translation becomes

$$\begin{array}{rccl} \lambda\colon & G & \longrightarrow & \mathrm{Perm}(G), \\ & g & \longmapsto & \lambda(g)(h) = gh, \end{array}$$

which is nothing but the left regular representation of G into $\mathrm{Perm}(G)$. Thus, N being G-stable is just the condition that N is normalized by $\lambda(G)$.

The absence of G' allows us to consider an analogous map by the right side.

Definition 2.4.18. *Let L/K be a Galois extension with group G. The right regular representation of L/K is defined as the one of G, that is,*

$$\begin{array}{rccl} \rho\colon & G & \longrightarrow & \mathrm{Perm}(G), \\ & g & \longmapsto & \rho(g)(h) = hg^{-1}. \end{array}$$

The right regular representation ρ is clearly injective, as in the case of λ. In fact, $\rho(G)$ is the group of the right translations. Under this language, we have the following.

Proposition 2.4.19. *Let G be a group.*

1. *$\lambda(G)$ and $\rho(G)$ are regular subgroups of* $\mathrm{Perm}(G)$.
2. *$\rho(G)$ is centralized by $\lambda(G)$.*
3. *$\rho(G) = \lambda(G)$ if and only if G is abelian.*

As a consequence, $\lambda(G)$ and $\rho(G)$ are regular and G-stable subgroups, therefore giving Hopf-Galois structures on L/K.

Proposition 2.4.20 ([Chi00], (6.10))**.** *Let L/K be a Galois extension with group G. Then $\rho(G)$, as a regular and G-stable subgroup of* $\mathrm{Perm}(G)$*, corresponds to the classical Galois structure $(K[G], \cdot)$ on L/K.*

By Proposition 2.4.19 3, when G is abelian, $\lambda(G)$ and $\rho(G)$ give the same Hopf-Galois structure; otherwise they give two different Hopf-Galois structures.

Definition 2.4.21. *Let L/K be a Galois extension with group G and suppose that G is not abelian. The Hopf-Galois structure on L/K corresponding to $\lambda(G)$ is called the* ***canonical non-classical structure****.*

When both Hopf-Galois structures arise, we shall use the label H_c for the classical Galois structure, and write H_λ for the canonical non-classical structure.

4.4 An example of application

Let $L = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $f(x) = x^3 - 3x + 3$. Let us find all the Hopf-Galois structures on $L/\mathbb{Q}$ using Greither-Pareigis theorem.

First, we identify the groups G and G'. Since $[L : K] = 3$, G can be embedded as a transitive subgroup of $S_3 = D_3$, namely, $G \cong C_3$ or $G \cong D_3$. Since the discriminant of f is $\mathrm{disc}(f) = -135 = -3^3 \cdot 5$, which is not a square, we obtain that $G \cong D_3$. Therefore, G can be presented as

$$G = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1_G,\ \tau\sigma = \sigma^2\tau \rangle.$$

Under the Galois correspondence, L maps to $G' = \mathrm{Gal}(\widetilde{L}/L)$, and since $[L : K] = [G : G']$, G' is an order 2 subgroup of G. The order 2 subgroups of G are $\langle\tau\rangle$, $\langle\sigma\tau\rangle$ and $\langle\sigma^2\tau\rangle$; we can assume without loss of generality that $G' = \{\tau\}$.

Let us describe how σ and τ act on $\widetilde{L}$. We have $\widetilde{L} = \mathbb{Q}(\alpha, z)$ for $z = \sqrt{-15}$, so it is enough to give the images of α and z (since the definition in the other elements of $\widetilde{L}$ is given by extension by $\mathbb{Q}$-linearity). We know that σ can be seen as a permutation of the roots of f, so $\sigma(\alpha)$ is just one of the other two roots of f. This would give two possibilities for σ, among which there is free choice (exchange of the other two roots of f means replacement of σ by σ^2). On the other hand, let $M = \mathbb{Q}(z)$, which is a subfield of $\widetilde{L}$ that is quadratic over $\mathbb{Q}$. Then, under the Galois correspondence it yields an order 3 subgroup of G, but the only one is $\langle\sigma\rangle$. Therefore, $\mathrm{Gal}(\widetilde{L}/M) = \langle\sigma\rangle$, whence $\sigma(z) = z$. As for τ, the equality $G' = \langle\tau\rangle$ gives $\tau(\alpha) = \alpha$, and $\tau(z) = -z$ follows from the fact that $z^2 \in \mathbb{Q}$.

By Greither-Pareigis theorem, the Hopf-Galois structures on $L/\mathbb{Q}$ are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G/G')$ normalized by $\lambda(G)$. We have

$$G/G' = \{\overline{1_G}, \overline{\sigma}, \overline{\sigma^2}\}, \quad \overline{\sigma^i} = \{\sigma^i, \sigma^i\tau\},\ i = 0, 1, 2.$$

On the other hand, the left translation map of L/K is the map $\lambda\colon G \longrightarrow \mathrm{Perm}(G/G')$ defined by $\lambda(\sigma^i)(\overline{\sigma^j}) = \overline{\sigma^{i+j}}$.

Let us find the regular subgroups of $\mathrm{Perm}(G/G')$, which in particular have order 3. Since $|G/G'| = 3$, $\mathrm{Perm}(G/G') \cong S_3 = D_3$, the dihedral group of order 6. This possesses a unique order 3 subgroup

$$N \coloneqq \{\mathrm{Id}_{G/G'}, (\overline{1_G}, \overline{\sigma}, \overline{\sigma^2}), (\overline{1_G}, \overline{\sigma^2}, \overline{\sigma})\}.$$

This is regular, as it is easy to check that its action on G/G' is transitive.

Thus, N defined as above is the only regular subgroup of $\mathrm{Perm}(G/G')$. Note that $N = \lambda(J)$, and this is normalized by $\lambda(G)$ because J is a normal subgroup of G and λ is injective. Therefore, N is the only regular and G-stable subgroup of $\mathrm{Perm}(G/G')$, and hence, L/K admits a unique Hopf-Galois structure H. Let us determine it.

We begin with the underlying Hopf algebra. Let $\cdot$ be the action of G on $\widetilde{L}[N]$ given by the classical Galois action on $\widetilde{L}$ and by conjugation with $\lambda(G)$ on N. Then, the underlying Hopf algebra H is formed by the elements of $\widetilde{L}[N]$ that are fixed by this action. Pick $h \in \widetilde{L}[N]^G$, so $h = \sum_{i=0}^{2} a_i \lambda(\sigma^i)$ for some $a_i \in \widetilde{L}$ and $g \cdot h = h$ for all $g \in G$. It is enough to study the action of the generators σ and τ of G. We have that

$$\sigma * \lambda(\sigma^i) = \lambda(\sigma\sigma^i\sigma^{-1}) = \lambda(\sigma^i), \quad i = 1, 2, 3,$$

so

$$h = \sigma * h = \sum_{i=0}^{2} \sigma(a_i)\lambda(\sigma^i).$$

By the uniqueness of coordinates, $a_i = \sigma(a_i)$ for all i, whence $a_i \in \widetilde{L}^{\langle\sigma\rangle} = M$. On the other hand,

$$\tau * \lambda(\sigma^i) = \lambda(\tau\sigma^i\tau^{-1}) = \lambda(\sigma^{-i}), \quad i = 1, 2, 3,$$

whence

$$h = \tau * h = \tau(a_0)1_{G/G'} + \tau(a_2)\lambda(\sigma) + \tau(a_1)\lambda(\sigma^2).$$

We deduce that $a_0 \in \widetilde{L}^{\langle\tau\rangle} = L$, so $a_0 \in L \cap M = \mathbb{Q}$, and $\tau(a_1) = a_2$, $\tau(a_2) = a_1$ (even though the second equality is redundant because τ is of order 2). Since $a_1 \in M = \mathbb{Q}(z)$, there are $b, c \in \mathbb{Q}$ such that $a_1 = b + cz$. Applying τ we obtain that $a_2 = b - cz$. Let us relabel $a_0 = a$. Then

$$\begin{aligned} h &= a_0\mathrm{Id}_{G/G'} + a_1\lambda(\sigma) + a_2\lambda(\sigma^2) \\ &= a\mathrm{Id}_{G/G'} + (b + cz)\lambda(\sigma) + (b - cz)\lambda(\sigma^2) \\ &= a\mathrm{Id}_{G/G'} + b(\lambda(\sigma) + \lambda(\sigma^2)) + cz(\lambda(\sigma) - \lambda(\sigma^2)) \end{aligned}$$

Hence, h lies in the subspace of $\widetilde{L}[N]$ generated by $1_{G/G'}$, $\lambda(\sigma) + \lambda(\sigma^2)$ and $z(\lambda(\sigma) - \lambda(\sigma^2))$. Since $h \in H$ is arbitrary, H is contained in such a subspace. But both H and the subspace have dimension 3 over $\mathbb{Q}$, so they coincide. In other words, H has $\mathbb{Q}$-basis

$$\{1_{G/G'}, \lambda(\sigma) + \lambda(\sigma^2), z(\lambda(\sigma) - \lambda(\sigma^2))\}.$$

Finally, let us determine the action of H on L. Of course, it is enough to find it on the basis elements of H, and for $1_{G/G'}$, it is trivial. Therefore, we are left to find how $\lambda(\sigma) + \lambda(\sigma^2)$ and $z(\lambda(\sigma) - \lambda(\sigma^2))$ act on elements of L. Given $x \in L$,

$$(\lambda(\sigma) + \lambda(\sigma^2)) \cdot x = \lambda(\sigma)^{-1}(\mathrm{Id}_G)(x) + \lambda(\sigma^2)^{-1}(\mathrm{Id}_G)(x) = \sigma^2(x) + \sigma(x) = (\sigma + \sigma^2)(x),$$

$$z(\lambda(\sigma) - \lambda(\sigma^2)) \cdot x = z(\sigma^2(x) - \sigma(x)) = -z(\sigma - \sigma^2)(x).$$

5 Further applications of Greither-Pareigis theory

5.1 Almost classically Galois extensions revisited

In Section 3.1 we introduced the class of finite separable extensions L/K for which one can find a normal extension M/K such that $L \cap M = K$ and the compositum of L and M is just the normal closure $\widetilde{L}$ of L/K. These extensions are usually called **almost classically Galois** in literature. Furthermore, it has been shown in Proposition 2.3.1 that such extensions are Hopf-Galois. In this section we consider them under the reformulation introduced in Section 4 and deepen in their properties.

First, let us view a notion that arises in the situation of an almost classically Galois extension, which we will find very often in the sequel.

Definition 2.5.1. *Let K be a field and let L and M be field extensions of K with $L, M \subset \overline{K}$. We say that L/K and M/K are linearly disjoint (or that L and M are K-linearly disjoint) if the map*

$$\begin{array}{ccc} L \otimes_K M & \longrightarrow & LM \\ x \otimes y & \longmapsto & xy \end{array}$$

is an isomorphism of K-algebras.

Note that the map at Definition 2.5.1 is always an epimorphism of K-algebras. The fact that two field extensions L/K, M/K are linearly disjoint means that for $x, x' \in L$ and $y, y' \in M$, $xy = x'y'$ if and only if there is some non-zero $r \in K$ such that $x' = rx$ and $y = ry'$ (actually, the latter is implied by the former). The intuition is that at the compositum of L and M, no elements of either field are collapsed. This phenomenon can be visualized through the following result:

Proposition 2.5.2. *If two field extensions L/K and M/K are linearly disjoint, then $L \cap M = K$. Moreover, if either of the extensions is separable and either (possibly the same) normal, the converse holds.*

Proof. The first part is easy and left as exercise. For the converse, see [Coh91, Chapter 5, Theorem 5.5]. □

In particular, two extensions L/K and M/K with M/K Galois are linearly disjoint if and only if $L \cap M = K$.

We study several equivalent definitions of an almost classically Galois extension.

Theorem 2.5.3. *Let L/K be a (G, G')-separable extension. The following statements are equivalent:*

1. *L/K is almost classically Galois.*
2. *There is some finite and Galois extension M/K such that $L \otimes_K M \cong \widetilde{L}$ as K-algebras.*
3. *There is some finite and Galois extension M/K such that $L \otimes_K M$ is isomorphic as a K-algebra to a field containing $\widetilde{L}$.*
4. *There is some normal complement J for G' in G.*
5. *There is a regular and G-stable subgroup N of $\mathrm{Perm}(G/G')$ such that $N \subset \lambda(G)$, where $\lambda\colon G \longrightarrow \mathrm{Perm}(G/G')$ is the left translation map of L/K.*

Proof. Suppose that L/K is almost classically Galois, so that there is a Galois extension M/K such that $L \cap M = K$ and $LM = \widetilde{L}$. Since M/K is Galois, from Proposition 2.5.2 we see that $L \otimes_K M \cong \widetilde{L}$ as K-algebras. Conversely, assume that there is a Galois extension M/K such that $L \otimes_K M \cong \widetilde{L}$ as K-algebras; in particular, $L \otimes_K M$ is a field. Taking into account the definition of the multiplication at $L \otimes_K M$, necessarily $L \otimes_K M \cong LM$ as K-algebras. Together with the previous isomorphism, we obtain $\widetilde{L} = LM$ and the map at Definition 2.5.1 is an isomorphism of K-algebras, so L/K is almost classically Galois.

It is trivial that 2 implies 3. Let us prove the converse. Let M/K be a Galois extension such that $L \otimes_K M$ is a field and $\widetilde{L} \hookrightarrow L \otimes_K M$ as K-algebras. We shall prove that M/K can be shrunk to a Galois extension M'/K such that $L \otimes_K M' \cong \widetilde{L}$ as K-algebras (see [GP87, Proof of Theorem 2.5]). Since $L \otimes_K M$ is a field, arguing as above, $L \otimes_K M \cong LM$ as K-algebras, so $\widetilde{L} \subseteq LM$. Now, by definition, $\widetilde{L}M$ is a field containing L and M, so $LM \subseteq \widetilde{L}M$. Joining both inclusions, we have $LM = \widetilde{L}M$, proving that LM/K is Galois. Hence, so are the extensions $LM/\widetilde{L}$, LM/L and LM/M. Call $\Gamma = \mathrm{Gal}(LM/K)$, $\overline{\Gamma} = \mathrm{Gal}(LM/\widetilde{L})$, $\Gamma_L = \mathrm{Gal}(LM/L)$ and $\Gamma_M = \mathrm{Gal}(LM/M)$. Since the lattice of subgroups of Γ is distributive with respect to the product and the intersection of subgroups and $\overline{\Gamma} \subseteq \Gamma_L$, $\overline{\Gamma}(\Gamma_L \cap \Gamma_M) = (\overline{\Gamma} \cdot \Gamma_L) \cap (\overline{\Gamma} \cdot \Gamma_M) = \Gamma_L \cap (\overline{\Gamma} \cdot \Gamma_M)$. Since the Galois correspondence is inclusion-reversing, applying it at both sides of the equality yields

$$\widetilde{L} \cap (LM) = L(\widetilde{L} \cap M).$$

But recall that LM contains $\widetilde{L}$, so $\widetilde{L} = \widetilde{L} \cap (LM) = L(\widetilde{L} \cap M)$. Let us define $M' := \widetilde{L} \cap M$. Since $\widetilde{L}/K$ and M/K are Galois, so is M'/K. Moreover, the previous equality becomes $\widetilde{L} = LM'$. It remains to prove that $LM' \cong L \otimes_K M'$ as K-algebras, or equivalently, that L and M' are K-linearly disjoint. Since $LM \cong L \otimes_K M$ as K-algebras, $L \cap M = K$. Moreover $M' \subseteq M$, so $L \cap M' = K$. Given that M'/K is Galois, applying Proposition 2.5.2 we obtain that L/K and M'/K are linearly disjoint, as we wanted.

Let us show that 1 and 4 are equivalent. Let M be an intermediate field of $\widetilde{L}/K$ and let $J = \mathrm{Gal}(\widetilde{L}/M)$. By the fundamental theorem of Galois theory, M/K is Galois if and only if J is a normal subgroup of G. In addition, $L \cap M = K$ and $LM = \widetilde{L}$ if and only if $JG = G'$ and $J \cap G' = \{1_G\}$. Hence, L/K is almost classically Galois with complement M if and only if J is a normal complement for G' in G, as we wanted.

Finally, we show the equivalence between 4 and 5. Suppose that there is a normal complement J for G' in G and let $N := \lambda(J)$. Since J is a normal subgroup of G and λ is a group monomorphism, N is G-stable. Let us see that the map $\varphi_{\overline{1}} \colon N \longrightarrow G/G'$ defined by $\varphi_{\overline{1}}(\eta) = \eta(\overline{1})$ is bijective. For $\sigma \in J$, $\varphi_{\overline{1}}(\lambda(\sigma)) = \lambda(\sigma)(\overline{1}) = \overline{\sigma}$. Since $G = JG'$ and $J \cap G'$, for each $g \in G$ there are unique $\sigma \in J$ and $\tau \in G'$ such that $g = \sigma\tau$, so $\overline{g} = \overline{\sigma\tau} = \overline{\sigma}$. Hence, each left coset in G/G' admits as representative a unique element of J (we say that J is a transversal of G' in G). This proves that $\varphi_{\overline{1}}$ is surjective, and the bijectivity follows from $|N| = |G/G'|$. By Proposition 2.4.2, N is regular.

Conversely, suppose that there is a regular and G-stable subgroup N of $\mathrm{Perm}(G/G')$ with $N \subset \lambda(G)$. Call $J := \lambda^{-1}(N)$. Since N is G-stable, J is a normal subgroup of G. First, let us note that for each $\tau \in G'$, $\lambda(\tau)(\overline{1_G}) = \overline{\tau} = \overline{1_G}$. Then $\lambda(G') \subset \mathrm{Stab}_{\lambda(G)}(\overline{1})$.

Now, since N is regular, we have that $\mathrm{Stab}_N(\overline{1}) = \{1_N\}$. Hence,

$$N \cap \lambda(G') \subset N \cap \mathrm{Stab}_{\lambda(G)}(\overline{1}) = \mathrm{Stab}_N(\overline{1}) = \{1_N\}.$$

We deduce that $N \cap \lambda(G') = \{1_N\}$. Applying λ^{-1}, we obtain $J \cap G' = \{1_G\}$. On the other hand, we have that $N\lambda(G') \subseteq \lambda(G)$ and

$$|N\lambda(G')| = |N|\,|\lambda(G')| = |G/G'|\,|G'| = |G| = |\lambda(G)|,$$

so $N\lambda(G') = \lambda(G)$. Applying λ^{-1}, we get $JG' = G$. We conclude that J is a normal complement for G' in G. □

For an almost classically Galois extension L/K with normal closure $\widetilde{L}$, $G = \mathrm{Gal}(\widetilde{L}/K)$ and $G' = \mathrm{Gal}(\widetilde{L}/L)$, we will say that L/K is (G, G')-almost classically Galois.

Remark 2.5.4. From the proof of Proposition 2.5.3 we can see that a field M satisfies 1 if and only if it satisfies 2, and that a field satisfying either is contained in a field satisfying 3. Moreover, a subgroup J of G satisfies 4 if and only if $\lambda(J)$ satisfies 5.

We have also seen that $\lambda(G') \subseteq \mathrm{Stab}_{\lambda(J)}(\overline{1_G})$, where the stabilizer corresponds to the group action of $\lambda(G)$ on G/G' by evaluation. Since λ is an injection, we can carry this to an action of G on G/G'. In this context, we actually prove the equality.

Corollary 2.5.5. *Let L/K be an (G, G')-almost classically Galois extension and let J be a normal complement for G' in G. Consider the action of G on $\mathrm{Perm}(G/G')$ induced by λ. For $N = \lambda(J)$, we have*

$$G' = \mathrm{Stab}_N(\overline{1}) \equiv \{g \in G \mid \lambda(g)(\overline{1}) = \overline{1}\}.$$

Proof. The action of G on $\mathrm{Perm}(G/G')$ is defined as follows: for $g \in G$ and $\eta \in N$, $g(\eta) = \lambda(g)(\eta)$. Now, for $g \in G$, $g(\overline{1}) = \overline{g}$, so $g \in \mathrm{Stab}_G(\overline{1})$ if and only if $\overline{g} = \overline{1}$; if and only if $g \in G'$. □

It is also possible to define a notion of almost classically Galois structure.

Let L/K be a (G, G')-separable almost classically Galois extension. By Theorem 2.5.3 5, there is some subgroup N giving a Hopf-Galois structure on L/K under the Greither-Pareigis correspondence that in addition satisfies $N \subset \lambda(G)$. However, L/K might admit other Hopf-Galois structures, and so, given by subgroups that lie outside $\lambda(G)$. We give a name to those Hopf-Galois structures that come from a normal complement.

Definition 2.5.6. *Let L/K be a (G, G')-almost classically Galois extension. We say that a Hopf-Galois structure on L/K is almost classically Galois if its corresponding subgroup of $\mathrm{Perm}(G/G')$ under the Greither-Pareigis correspondence satisfies $N \subset \lambda(G)$.*

We have from Theorem 2.5.3 5 that every almost classically Galois extension admits some almost classically Galois structure H. Let N be the corresponding permutation subgroup, so that $N \subset \lambda(G)$. Since λ is a group embedding, we have that $N = \lambda(J)$ for some normal subgroup J of G. This may be a normal complement of G', but not necessarily.

Example 2.5.7. Let L/K be a (G, G')-separable extension with $G \cong D_4$ and $G' \cong C_2$. Then L/K is almost classically Galois because $G = J \rtimes G'$ with $J \cong C_4$. Call $J = \langle \sigma \mid \sigma^4 = 1_G \rangle$ and $G' = \langle \tau \mid \tau^2 = 1_G \rangle$, so that

$$G = \langle \sigma, \tau \mid \sigma^4 = 1_G,\ \tau^2 = 1_G,\ \tau\sigma = \sigma^3\tau \rangle.$$

It can be checked that the regular and G-stable subgroups of $\mathrm{Perm}(G/G')$ are:

$$N = \langle (\overline{1}, \overline{\sigma}, \overline{\sigma^2}, \overline{\sigma^3}) \rangle,$$

$$N' = \langle (\overline{1}, \overline{\sigma})(\overline{\sigma^2}, \overline{\sigma^3}), (\overline{1}, \overline{\sigma^2})(\overline{\sigma}, \overline{\sigma^3}) \rangle.$$

Note that $N = \lambda(J)$ and $N' = \lambda(J')$ with $J' = \langle \sigma^2, \sigma\tau \rangle$. Both J and J' are normal subgroups, but only J serves as a normal complement for G', as $J' \cong C_2 \times C_2$.

Note that an almost classically Galois structure need not be unique, just because a normal complement for a subgroup G' of a group G is not unique in general.

The underlying Hopf algebra of an almost classically Galois structure admits a simpler expression than the one given at Proposition 2.4.16 1.

Proposition 2.5.8. *Let L/K be a (G, G')-almost classically Galois extension with complement M. Let N be a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$ with $N \subset \lambda(G)$. Then $\widetilde{L}[N]^G = M[N]^{G'}$.*

Proof. Recall that the action of G on $\widetilde{L}[N]$ is the one given at (2.1). Let $J = \mathrm{Gal}(\widetilde{L}/M)$. Since $G = J \rtimes G$, we have $\widetilde{L}[N]^G = (\widetilde{L}[N]^J)^{G'}$. Since $N \subset \lambda(G)$, $N = \lambda(J')$ for some normal subgroup J' of G. Since the conjugation by J leaves J' invariant, the conjugation by $\lambda(J)$ leaves $\lambda(J') = N$ invariant. Therefore, $\widetilde{L}[N]^J = \widetilde{L}^J[N] = M[N]$, and the statement follows. □

Example 2.5.9. Let us go back to the example from Section 4.4. We saw that $J = \langle \sigma \rangle$ is a normal complement for $G' = \langle \tau \rangle$, so L/K is almost classically Galois. When we picked an element $h \in \widetilde{L}[N]^G$, the condition that it is fixed by the action of J lead that it belongs to $M[N]^{G'}$, because h is already fixed by such an action. This is because $N = \lambda(J)$, so the action of J by conjugation leaves N invariant. One can see that the basis elements that we obtained are indeed fixed by the action of G'.

5.2 Byott's uniqueness theorem

In this part we study a sufficient condition found by Byott in the paper [Byo96] so as to ensure that a separable Hopf-Galois extension admits a unique Hopf-Galois structure, which is established using the techniques from Byott's translation that we saw at Section 3.2. Such a condition is related with a class of integer numbers, that are called Burnside.

Definition 2.5.10. *Let n be an integer number. We say that n is Burnside if it is coprime with its image by the Euler totient function φ, that is, $\gcd(n, \varphi(n)) = 1$.*

It is trivial from the definition that every prime number is Burnside. Moreover, every Burnside number is square-free. This follows directly from the remarks that $\varphi(p^r) = p^{r-1}(p-1)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ if $\gcd(a, b) = 1$. Burnside numbers are linked with group theory by the following result.

Theorem 2.5.11 (Burnside). *Let $n \in \mathbb{Z}_{\geq 1}$ be a positive integer. Then every group of order n is cyclic if and only if n is Burnside.*

Byott's uniqueness thereom provides a sufficient condition which ensures that a separable Hopf-Galois extension admits a unique Hopf-Galois structure. Namely, this condition is that the degree of the extension is a Burnside number. We need the following technical lemma.

Lemma 2.5.12. *Let L/K be a (G, G')-separable degree n extension and suppose that n is Burnside. Suppose that N is a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$ and let Λ_N be the set of left translations $\lambda_\eta \colon N \longrightarrow N$, $\eta \in N$. For each subgroup H of $\mathrm{Hol}(N)$ whose order is divisible by n, $\Lambda_N \subset H$.*

Proof. Recall that $\mathrm{Hol}(N) = \Lambda_N \rtimes \mathrm{Aut}(N)$ by definition. Consider the projection $\pi_2 \colon \mathrm{Hol}(N) \longrightarrow \mathrm{Aut}(N)$ onto the second component. Since π_2 is a group epimorphism (because $\mathrm{Aut}(N) \cong \mathrm{Hol}(N)/\Lambda_N$), it maps H onto a subgroup of $\mathrm{Aut}(N)$, whose order divides the order of $\mathrm{Aut}(N)$. Now, since n is Burnside and N has order n, we have that $N \cong \mathbb{Z}/n\mathbb{Z}$, and hence $\mathrm{Aut}(N) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. We deduce that $\mathrm{Aut}(N)$ has order $\varphi(n)$. It follows that the order of $\pi_2(H)$ divides $\varphi(n)$. Now, $\pi_2(H) \cong H\Lambda_N/\Lambda_N \cong H/\Lambda_N \cap H$, whence $|H/\Lambda_N \cap H| = \frac{|H|}{|\Lambda_N \cap H|}$ divides $\varphi(n)$. Taking into account that n divides $|H|$ and $\gcd(n, \varphi(n)) = 1$, necessarily $|\Lambda_N \cap H|$ is divisible by n. By the structure of semidirect product, Λ_N is the only order n subgroup of $\mathrm{Hol}(N)$. Now, Cauchy theorem gives that $\Lambda_N \cap H$ must have some subgroup of order n, which is necessarily Λ_N. We get $\Lambda_N \cap H = \Lambda_N$, and hence $\Lambda_N \subset H$ follows. □

Theorem 2.5.13 ([Byo96], Theorem 2). *Let L/K be a G-separable degree n extension. If L/K is Hopf-Galois and n is Burnside, then G is solvable and L/K admits a unique Hopf-Galois structure, which is almost classically Galois (in particular, L/K is almost classically Galois).*

Proof. Let $\widetilde{L}$ be the normal closure of L/K, $G = \mathrm{Gal}(\widetilde{L}/K)$ and $G' = \mathrm{Gal}(\widetilde{L}/L)$. The hypothesis that L/K is Hopf-Galois ensures that it admits some Hopf-Galois structure H; let N be its corresponding regular and G-stable subgroup of $\mathrm{Perm}(G/G')$. Let $\alpha \colon N \longrightarrow \mathrm{Perm}(G/G')$ be the canonical inclusion. By Theorem 2.3.7, α corresponds to a group embedding $\beta \colon G \longrightarrow \mathrm{Perm}(N)$ such that $\beta(G) \subset \mathrm{Hol}(N)$. Now, note that $\Lambda_N = \lambda_N(N)$, where $\lambda_N \colon N \longrightarrow \mathrm{Aut}(N)$ is the left regular representation of N. Since λ_N is injective, Λ_N has order n. By Theorem 2.5.11, both Λ_N and N are cyclic. In addition, $\mathrm{Aut}(N) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, which is abelian. Therefore, $\mathrm{Hol}(N)$ is solvable. Since $\beta(G) \subset \mathrm{Hol}(N)$ with β injective, we conclude that G is solvable.

Let us prove that H is an almost classically Galois structure on L/K. We have that $\beta(G)$ is a subgroup of $\mathrm{Hol}(N)$ and its order is that of G, which is a multiple of n. Applying Lemma 2.5.12 with $H = \beta(G)$, we get $\Lambda_N \subset \beta(G)$. Going over the proof of Theorem 2.3.7, we see by (a) that $\beta = Ca^{-1} \circ \lambda$, where $Ca \colon \mathrm{Perm}(N) \longrightarrow \mathrm{Perm}(G/G')$ is the group isomorphism induced by the bijection $a \colon N \longrightarrow G/G'$, $a(\eta) = \eta(\overline{1_G})$. On the other hand, in (b) it is shown that $\lambda_N = Ca^{-1} \circ \alpha$. Applying Ca on the previous inclusion, we obtain that $N = \alpha(N) \subset \lambda(G)$, so the Hopf-Galois structure corresponding to N is almost classically Galois.

Finally, we shall prove that L/K does not admit other Hopf-Galois structures. Suppose that N' is a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$. If we consider

the canonical inclusion $\alpha' \colon N' \longrightarrow \mathrm{Perm}(G/G')$, the definition of α and α' are the same. Thus, if $\beta' \colon G \hookrightarrow \mathrm{Hol}(N')$ is the group embedding corresponding to α' by Byott's theorem, we have that $\beta' = Ca'^{-1} \circ \alpha'$, where $Ca' \colon \mathrm{Perm}(N') \longrightarrow \mathrm{Perm}(G/G')$ is the group isomorphism induced by the bijection $a \colon N' \longrightarrow G/G'$, and then the definitions of β and β' are the same. Then we can regard $\beta(G)$ as a subgroup of $\mathrm{Hol}(N')$. We then apply Lemma 2.5.12 with N' as regular and G-stable subgroup and $H = \beta(G)$, obtaining that $\Lambda_{N'} \subset \beta(G) \subset \mathrm{Hol}(N)$. Hence $\Lambda_{N'}$ is an order n subgroup of $\mathrm{Hol}(N)$, so once again by Lemma 2.5.12 (with N as regular and G-stable subgroup and $H = \Lambda_{N'}$), we obtain $\Lambda_N \subset \Lambda_{N'}$, both of which have order n. Necessarily $\Lambda_N = \Lambda_{N'}$, that is, $\lambda_N(N) = \lambda_{N'}(N')$. We use again that α and α' have the same definition to obtain that $\lambda_N = Ca^{-1} \circ \alpha$ and $\lambda_{N'} = Ca'^{-1} \circ \alpha'$ also do, to conclude that $N = N'$. □

A G-separable degree n extension with n Burnside and G solvable is not necessarily Hopf-Galois (see [Byo96, Example after Theorem 2] for a counterexample). Then, Theorem 2.5.13 can be restated by saying that a separable degree n extension with n Burnside admits at most one Hopf-Galois structure.

We shall show that the converse of Theorem 2.5.13 does not hold: a G-separable extension with G solvable and admitting a unique Hopf-Galois structure has not necessarily Burnside degree. Indeed, it can be checked that a G-separable quartic extension with $G \cong A_4$ or $G \cong S_4$ (thus, G solvable) admits a unique Hopf-Galois structure, and 4 is not a Burnside number because $\varphi(4) = 2$.

Let us specify Theorem 2.5.13 to the Galois case. If L/K is a Galois extension with group G with Burnside degree, then G is solvable and the classical Galois structure on L/K is the unique Hopf-Galois structure on L/K. In this case, having Burnside degree becomes a characterization for the uniqueness of Hopf-Galois structure.

Theorem 2.5.14 ([Byo96], Theorem 1)**.** *A degree n Galois extension L/K admits a unique Hopf-Galois structure if and only if n is Burnside.*

The right-to-left implication is just a particular case of Theorem 2.5.13. A proof for the converse can be found in [Chi00, §8].

Since prime numbers are Burnside, Theorem 2.5.13 yields the following.

Corollary 2.5.15. *Let p be a prime number and let L/K be a Hopf-Galois extension with degree p. Then L/K admits a unique Hopf-Galois structure.*

5.3 Opposite Hopf-Galois structures

The Greither-Pareigis theorem establishes a connection between the Hopf-Galois structures on a separable extension and group theory. Thus, one can wonder if notions or results on the latter can be translated to the former. One of these is the notion of opposite group, of which we shall study its Hopf-Galois counterpart. For a group $(N, \star)$, its opposite, denoted by N^{opp}, is defined as the group whose underlying set is also N and the operation is defined as

$$\eta \star' \mu := \mu \star \eta, \quad \mu, \eta \in N.$$

Let N be a permutation subgroup giving a Hopf-Galois structure on a separable extension under the Greither-Pareigis correspondence. We shall see that N^{opp} also

gives a Hopf-Galois structure on the same extension. But, in order to do so, we need to visualize it as a subgroup of the same permutation group. We see that we can identify it with the centralizer of N.

Proposition 2.5.16. *Let X be a finite set and let N be a regular subgroup of $\mathrm{Perm}(X)$. Fix $x_0 \in X$ and for each $\eta \in N$, define a map $\phi_\eta \colon X \longrightarrow X$ as follows: for $x \in X$, $\phi_\eta(x) = \mu_x \circ \eta(x_0)$, where $\mu_x \in N$ is such that $\mu_x(x_0) = x$. The following statements hold:*

1. *For every $\eta \in N$, ϕ_η is well defined and bijective.*
2. *$\mathrm{Cent}_{\mathrm{Perm}(X)}(N) = \{\phi_\eta \mid \eta \in N\}$.*
3. *The map $\Phi \colon N^{\mathrm{opp}} \longrightarrow \mathrm{Cent}_{\mathrm{Perm}(X)}(N)$ defined by $\Phi(\eta) = \phi_\eta$ is a group isomorphism.*

Proof. 1. Let $\eta \in N$. Since N is regular, for each $x \in X$ there is a unique $\mu_x \in N$ such that $\mu_x(x_0) = x$. This ensures that ϕ_η is well defined. Let us see that it is bijective. Let $x, y \in X$ such that $\phi_\eta(x) = \phi_\eta(y)$, that is, $\mu_x \circ \eta(x_0) = \mu_y \circ \eta(x_0)$. Since $\mu_x, \mu_y, \eta \in N$, we have that both $\mu_x \circ \eta$ and $\mu_y \circ \eta$ belong to N, and by the regularity of N, they are completely determined by their definition at x_0. Hence, $\mu_x \circ \eta = \mu_y \circ \eta$, and composing on the right side by η^{-1}, we get $\mu_x = \mu_y$. Evaluating at x_0, we obtain that $x = y$, so ϕ_η is injective. Since it is defined from X to itself and X is finite, ϕ_η is bijective.

2. Let $\phi \in \mathrm{Cent}_{\mathrm{Perm}(X)}(N)$. By regularity, there is a unique $\eta \in N$ such that $\eta(x_0) = \phi(x_0)$. We claim that $\phi = \phi_\eta$. Take $x \in X$. By the definition of centralizer, $\mu_x \circ \phi = \phi \circ \mu_x$. Now,

$$\phi(x) = \phi \circ \mu_x(x_0) = \mu_x \circ \phi(x_0) = \mu_x \circ \eta(x_0) = \phi_\eta(x).$$

Hence $\phi = \phi_\eta$ as claimed. Conversely, take $\eta \in N$ and let us prove that ϕ_η centralizes N. We need to prove that, for each $\mu \in N$, $\phi_\eta \circ \mu = \mu \circ \phi_\eta$. Given $x \in X$, $\mu \circ \phi_\eta(x) = \mu \circ \mu_x \circ \eta(x_0)$. Now, note that

$$\mu \circ \mu_x(x_0) = \mu(x) = \mu_{\mu(x)}(x_0).$$

By regularity, $\mu \circ \mu_x = \mu_{\mu(x)}$. Then,

$$\mu \circ \phi_\eta(x) = \mu_{\mu(x)} \circ \eta(x_0) = \phi_\eta(\mu(x)) = \phi_\eta \circ \mu(x).$$

This proves that $\phi_\eta \circ \mu = \mu \circ \phi_\eta$, as we wanted.

3. We already know that for each $\phi \in \mathrm{Cent}_{\mathrm{Perm}(X)}(N)$ there is some η in the underlying set of N such that $\phi = \phi_\eta$. This is the same as the underlying set of N^{opp}, so Φ is surjective. On the other hand, if $\eta, \mu \in N$ are such that $\phi_\eta = \phi_\mu$, evaluating at any element $x \in X$ gives $\mu_x \circ \eta(x_0) = \mu_x \circ \mu(x_0)$, and composing by μ_x^{-1} on the left side gives $\eta(x_0) = \mu(x_0)$. Once again, the regularity of N gives that $\eta = \mu$. This proves that Φ is bijective. Let us check that it preserves the group structure. Given $\eta, \mu \in N$, we must check that $\Phi(\eta \circ' \mu) = \Phi(\eta) \circ \Phi(\mu)$, that is, $\phi_{\mu \circ \eta} = \phi_\eta \circ \phi_\mu$. Given $x \in X$, we have

$$\phi_\eta \circ \phi_\mu(x) = \mu_{\phi_\mu(x)} \circ \eta(x_0) = \mu_{\mu_x \circ \mu(x_0)} \circ \eta(x_0).$$

Now, $\mu_{\mu_x \circ \mu(x_0)}(x_0) = \mu_x \circ \mu(x_0)$, so $\mu_{\mu_x \circ \mu(x_0)} = \mu_x \circ \mu$ by regularity. Then,

$$\phi_\eta \circ \phi_\mu(x) = \mu_x \circ \mu \circ \eta(x_0) = \phi_{\mu \circ \eta}(x),$$

finishing the proof.

□

From now on, for each regular subgroup N of a permutation group $\mathrm{Perm}(X)$, we regard N^{opp} as a subgroup of $\mathrm{Perm}(X)$ by means of identifying $N^{\mathrm{opp}} = \mathrm{Cent}_{\mathrm{Perm}(X)}(N)$.

Proposition 2.5.17. *Let X be a finite set and let N be a regular subgroup of $\mathrm{Perm}(X)$. Then N^{opp} is regular.*

Proof. Since the underlying set of N^{opp} is the same as the underlying set of N, $|N^{\mathrm{opp}}| = |N|$. Let $x \in X$ and take $\phi \in \mathrm{Stab}_{N^{\mathrm{opp}}}(x)$. Let $\eta \in N$ be such that $\phi = \phi_\eta$. Then,

$$\mu_x \circ \eta(x_0) = \phi(x) = x = \mu_x(x_0).$$

The regularity of N yields that $\mu_x \circ \eta = \mu_x$, so $\eta = 1_N$. Then $\mathrm{Stab}_{N^{\mathrm{opp}}}(x) = \{\mathrm{Id}\}$. □

Now, we turn to the scenario of field extensions.

Proposition 2.5.18. *Let L/K be a (G, G')-separable extension and let N be a regular subgroup of $\mathrm{Perm}(X)$. If N is G-stable, then so is N^{opp}.*

Proof. Suppose that N is G-stable. Given $\eta \in N$ and $g \in G$, we shall prove that $\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1}) \in N$, that is, $\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1}) = \phi_{\eta'}$ for some $\eta' \in N$. Equivalently, $\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1})(x) = \mu_x \circ \eta'(x_0)$ for some $\eta' \in N$. We have that

$$\begin{aligned}
\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1})(x) &= \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta(x_0) \\
&= \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta \circ \lambda(g^{-1}) \circ \lambda(g)(x_0) \\
&= \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)}(x_0) \\
&= \mu_x \circ \mu_x^{-1} \circ \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)}(x_0).
\end{aligned}$$

Thus, it is enough to show that the element

$$\eta'_x := \mu_x^{-1} \circ \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)} \in N$$

does not depend on x. Note that

$$\lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)}(x_0) = x$$

with $\lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)} \in N$, so

$$\mu_x = \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)}.$$

Equivalently,

$$\lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \lambda(g^{-1}) = \mu_x \circ \mu^{-1}_{\lambda(g)(x_0)}.$$

Then,

$$\begin{aligned}\eta'_x &= \mu_x^{-1} \circ \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)} \\ &= \mu_x^{-1} \circ \lambda(g) \circ \mu_{\lambda(g^{-1})(x)} \circ \lambda(g^{-1}) \circ \lambda(g) \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)} \\ &= \mu_x^{-1} \circ \mu_x \circ \mu_{\lambda(g)(x_0)}^{-1} \circ \lambda(g) \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)} \\ &= \mu_{\lambda(g)(x_0)}^{-1} \circ \lambda(g) \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)},\end{aligned}$$

which does not depend on x_0. Let us relabel

$$\eta' \coloneqq \mu_{\lambda(g)(x_0)}^{-1} \circ \lambda(g) \circ \eta \circ \lambda(g^{-1}) \circ \mu_{\lambda(g)(x_0)}.$$

We obtain that

$$\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1})(x) = \mu_x \circ \eta'(x) = \phi_{\eta'}(x)$$

for every $x \in X$, whence $\lambda(g) \circ \phi_\eta \circ \lambda(g^{-1}) = \phi_{\eta'} \in N$. □

We conclude that, for a (G, G')-separable extension, the opposite of a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$ also is. The Greither-Pareigis correspondence yields the following notion.

Definition 2.5.19. *Let L/K be a (G, G')-separable Hopf-Galois extension. Let H be a Hopf-Galois structure on L/K and let N be a regular and G-stable subgroup of $\mathrm{Perm}(G/G')$. The **opposite Hopf-Galois structure** of H, denoted as H^{opp}, is the one whose corresponding permutation subgroup is N^{opp}.*

If N is abelian, the opposite Hopf-Galois structure of H is itself.

Let L/K be a Galois extension with group G and let λ (resp. ρ) be the left (resp. right) regular representation of G. Recall by Proposition 2.4.19 2 that $\rho(G)$ is centralized by $\lambda(G)$, whence $\mathrm{Cent}_{\mathrm{Perm}(G)}(\rho(G)) = \lambda(G)$. We obtain:

Corollary 2.5.20. *Let L/K be a Galois non-abelian extension. The opposite Hopf-Galois structure of the classical Galois structure is the canonical non-classical structure.*

Remark 2.5.21. The opposite of an almost classically Galois structure need not be almost classically Galois. As a counterexample, consider the situation at Corollary 2.5.20. The classical Galois structure on L/K corresponds to the subgroup $\lambda(G)$, while its opposite, the canonical non-classical structure, corresponds to $\rho(G)$. The classical Galois structure is trivially almost classically Galois, while the canonical non-classical structure is not because $\rho(G) \not\subset \lambda(G)$, which follows from Proposition 2.4.19 3.

Recall from Proposition 2.4.11 that if a Hopf-Galois structure H corresponds to a subgroup N, $\widetilde{L} \otimes_K H \cong \widetilde{L}[N]$ as $\widetilde{L}$-Hopf algebras. Using the notion of opposite group we can find the smallest field base field with that property.

Proposition 2.5.22. *Let L/K be a separable extension with normal closure $\widetilde{L}$. Let H be a Hopf-Galois structure on L/K and let N be its corresponding permutation subgroup. Let $G_0 = \lambda^{-1}(N^{\mathrm{opp}})$ and let $L_0 = \widetilde{L}^{G_0}$. Then L_0 is the smallest extension of K such that*

$$L_0 \otimes_K H \cong L_0[N]$$

as L_0-algebras.

The proof of Proposition 2.5.22 makes use of descent theory and cohomology, and it is beyond the scope of these notes. It can be consulted at [GP87, Corollary 3.2].

Note that G_0 is the subgroup of elements of G that fix all the elements of N by the action $*$ defined at (2.1). Assume that L/K is almost classically Galois with J a normal complement of G' and choose $N = \lambda(J)^{\mathrm{opp}}$. Then $N^{\mathrm{opp}} = \lambda(J)$ and, consequently, $J = G_0$. Thus, the field L_0 is the complement of L/K as an almost classically Galois extension.

Corollary 2.5.23. *Let L/K be a (G, G')-separable almost classically Galois extension with complement M, and let $J = \mathrm{Gal}(\widetilde{L}/M)$. Let $N = \lambda(J)^{\mathrm{opp}}$ and let H be its corresponding Hopf-Galois structure on L/K. Then M is the smallest extension of K such that*

$$M \otimes_K H \cong M[N].$$

It is remarkable that Corollary 2.5.23 states a property for the opposite of an almost classically Galois structure. For this reason, some authors call these almost classically Galois structures; namely, the ones given by a permutation subgroup N such that $N^{\mathrm{opp}} \subset \lambda(G)$.

5.4 Induced Hopf-Galois structures

Let E/K be a finite and Galois extension with Galois group of the form $G = J \rtimes G'$, where J is a normal subgroup of G and G' is any subgroup of G. Call $L = E^{G'}$. It is possible to build a Hopf-Galois structure on E/K from a pair of Hopf-Galois structures from E/L and L/K. Such Hopf-Galois structures are called induced, and were introduced by Crespo, Rio and Vela in the paper [CRV16].

In order to introduce the notion of induced Hopf-Galois structure, we make use of the Greither-Pareigis correspondence. Namely, we will see that the direct product of the permutation subgroups corresponding to Hopf-Galois structures on E/L and L/K is isomorphic to a subgroup giving a Hopf-Galois structure on L/K.

Both of the extensions E/K and E/L are Galois with groups G and G' respectively. By Greither-Pareigis theorem:

1. The Hopf-Galois structures on E/L are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G')$ normalized by the image of the left translation map $\lambda^{G'} \colon G' \longrightarrow \mathrm{Perm}(G')$.

2. The Hopf-Galois structures on E/K are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G)$ normalized by the image of the left translation map $\lambda \colon G \longrightarrow \mathrm{Perm}(G)$ of L/K.

The situation is a bit trickier for the extension L/K, which is typically non-Galois. Note that E is a Galois field extension of K containing L, so E contains the normal closure $\widetilde{L}$ of L. However, in general it does not hold that $E = \widetilde{L}$ (for instance, when the semidirect product is direct). By Proposition 2.4.3, we can apply Greither-Pareigis theorem to characterize the Hopf-Galois structures on L/K choosing any Galois extension of K containing L, not just its normal closure. In particular, we

can choose E/K. Thus, the Hopf-Galois structures on L/K are in bijective correspondence with the regular subgroups of $\mathrm{Perm}(G/G')$ normalized by $\lambda_E(G)$, where $\lambda_E\colon G \longrightarrow \mathrm{Perm}(G/G')$ is the left translation map of L/K associated to E.

Another of the key ingredients for the existence of induced Hopf-Galois structures is that J is a transversal of G' in G, which is a consequence of J being a normal complement for G' in G. This means that each left coset of G/G' intersects with J in exactly one element. Let us write $J = \{\sigma_1, \ldots, \sigma_n\}$, where $n = [L : K]$. Then, we can write $G/G' = \{\sigma_1 G', \ldots, \sigma_n G'\}$ and identify J with G/G'. Carrying this identification to the map $\lambda_L\colon G \longrightarrow \mathrm{Perm}(G/G')$, we get a map $\lambda_c\colon G \longrightarrow \mathrm{Perm}(J)$ whose definition corresponds to the action of G on the left cosets of G/G' by means of λ_L. Namely, for a given element $g \in G$, $\lambda_c(g)$ is the permutation of J that takes an element $\sigma_i \in J$ to the representative of J in the left coset $\lambda_L(g)(\sigma_i G')$. Let us calculate it. Write $g = \sigma\tau$ with $\sigma \in J$ and $\tau \in G'$. For every $1 \leq i \leq n$,

$$\begin{aligned}\lambda_L(g)(\sigma_i G') &= \sigma\tau\sigma_i G' \\ &= \sigma\tau\sigma_i\tau^{-1}\tau G' \\ &= \sigma C_\tau(\sigma_i) G' \\ &= \lambda^J(\sigma) \circ C_\tau(\sigma_i) G',\end{aligned}$$

where $C_\tau \in \mathrm{Aut}(G)$ is the conjugation by τ and $\lambda^J\colon J \longrightarrow \mathrm{Perm}(J)$ is the left translation map associated to L/F. Note that $C_\tau(\sigma_i) \in J$ because J is a normal subgroup of G, so $\lambda^J(\sigma) \circ C_\tau(\sigma_i)$ makes sense and belongs to J. Thus, for $g = \sigma\tau \in G$,

$$\lambda_c(g)(\sigma_i) = \lambda^J(\sigma) \circ C_\tau(\sigma_i) \tag{2.4}$$

Proposition 2.5.24. *Let $\lambda\colon G \longrightarrow \mathrm{Perm}(G)$ be the left regular representation of E/K. Then, $\lambda = \iota \circ \chi$, where*

$$\begin{array}{rccc} \chi\colon & G & \longrightarrow & \mathrm{Perm}(J) \times \mathrm{Perm}(G') \\ & \sigma\tau & \longmapsto & (\lambda_c(\sigma\tau), \lambda^{G'}(\tau)), \\ \iota\colon & \mathrm{Perm}(J) \times \mathrm{Perm}(G') & \longrightarrow & \mathrm{Perm}(G) \\ & (\varphi, \psi) & \longmapsto & \sigma\tau \mapsto \varphi(\sigma)\psi(\tau). \end{array}$$

Proof. Given $g = \sigma\tau \in G$ and $g' = \sigma'\tau' \in G$, we have

$$\begin{aligned}\lambda(g)(g') &= gg' = \sigma\tau\sigma'\tau' = \sigma\tau\sigma'\tau^{-1}\tau\tau' \\ &= \sigma C_\tau(\sigma')\lambda^{G'}(\tau)(\tau') \\ &= \lambda_c(g)(\sigma')\lambda^{G'}(\tau)(\tau') \\ &= \iota(\lambda_c(g), \lambda^{G'}(\tau))(g') \\ &= \iota \circ \chi(g)(g'),\end{aligned}$$

where, from the second to the third line, we have used (2.4). □

We are ready to build the so-called induced Hopf-Galois structures.

Proposition 2.5.25. *Let N_1 be a subgroup of $\mathrm{Perm}(J)$ and let N_2 be a subgroup of $\mathrm{Perm}(G')$. Let $N = \iota(N_1 \times N_2)$.*

1. *If N_1 and N_2 are regular, so is N.*

2. *If N_1 is normalized by $\lambda_L(G)$ and N_2 is normalized by $\lambda^{G'}(G')$, then N is normalized by $\lambda(G)$.*

Proof. 1. Suppose that N_1 and N_2 are regular. We have

$$|N| = |\iota(N_1 \times N_2)| = |N_1 \times N_2| = |N_1||N_2| = |J||G'| = |G|,$$

so it is enough to check that the action of N on G is transitive. Let $g = \sigma\tau$, $g' = \sigma'\tau' \in G$ with $\sigma, \sigma' \in J$ and $\tau, \tau' \in G'$. Since N_1 (resp. N_2) is regular, there exists $\varphi \in \mathrm{Perm}(J)$ (resp. $\psi \in \mathrm{Perm}(G')$) such that $\varphi(\sigma) = \sigma'$ (resp. $\psi(\tau) = \tau'$). Then,

$$\iota(\varphi, \psi)(g) = \iota(\varphi, \psi)(\sigma\tau) = \varphi(\sigma)\psi(\tau) = \sigma'\tau' = g'.$$

2. Let $g = \sigma\tau \in G$ with $\sigma \in J$ and $\tau \in G'$. Given $(\eta, \mu) \in N_1 \times N_2$,

$$\chi(g)(\eta, \mu)\chi(g^{-1}) = (\lambda_c(g)\eta\lambda_c(g)^{-1}, \lambda^{G'}(\tau)\mu\lambda^{G'}(\tau^{-1})) \in N_1 \times N_2.$$

Applying ι, since it is a group homomorphism, we obtain

$$\lambda(g)\iota(\eta, \mu)\lambda(g^{-1}) = \iota(\chi(g)(\eta, \mu)\chi(g^{-1})) \in \iota(N_1 \times N_2) = N.$$

□

Applying Greither-Pareigis theorem, we get the following.

Corollary 2.5.26. *Let E/K be a Galois extension with Galois group $G = J \rtimes G'$ and let $L = E^{G'}$. If N_1 is a subgroup of $\mathrm{Perm}(J)$ giving L/K a Hopf-Galois structure and N_2 is a subgroup of $\mathrm{Perm}(G')$ giving E/L a Hopf-Galois structure, then $N = \iota(N_1 \times N_2)$ is a subgroup giving E/K a Hopf-Galois structure.*

Definition 2.5.27. *A Hopf-Galois structure on E/K as in Corollary 2.5.26 is called an **induced Hopf-Galois structure** on E/K.*

If an induced Hopf-Galois structure H on E/K is built from Hopf-Galois structures H_1 on L/K and H_2 on E/L, we will also say that H is induced from H_1 and H_2, or that H_1 and H_2 induce H. The Hopf-Galois structures H_1 and H_2 receive the name of inducing Hopf-Galois structures.

Induced Hopf-Galois structures only make sense for Galois extensions whose Galois group is a semidirect product. In particular, if the Galois group of a Galois extension E/K is a direct product, then there are induced Hopf-Galois structures on E/K as well. In that case, both of the extensions E/L and L/K are Galois, and one can prove that the classical Galois structures on L/E and E/K induce the classical Galois structure on L/K.

We see an equivalent approach to think of induced Hopf-Galois structures.

Proposition 2.5.28. *Let E/K be a Galois extension with group $G = J \rtimes G'$ and call $L = E^{G'}$, $M = L^J$. Then, the Hopf Galois structures of E/L and M/K are in one-to-one correspondence.*

Proof. Let $\overline{G} := \mathrm{Gal}(M/K)$. Applying the Galois correspondence to $G = J \rtimes G'$, we get $L \cap M = K$, so the map $\cdot\,|_M \colon G' \longrightarrow \overline{G}$ defined by $\tau \mapsto \tau\,|_M$ is a group isomorphism. Moreover, E/L is Galois with group G', and since J is normal in G, M/K is Galois with group $\overline{G}$. By Greither-Pareigis theorem, the Hopf Galois structures of E/L (resp. M/K) are in one-to-one correspondence with the regular subgroups of $\mathrm{Perm}(G')$ (resp. $\mathrm{Perm}(\overline{G})$) normalized by $\lambda^{G'}(G')$ (resp. $\lambda^{\overline{G}}(\overline{G})$). The map $\cdot\,|_F$ induces a group isomorphism $\varphi \colon \mathrm{Perm}(G') \longrightarrow \mathrm{Perm}(\overline{G})$ defined as $\varphi(\eta) = \cdot\,|_F \circ \eta \circ (\cdot\,|_F)^{-1}$.

Let N be a subgroup of $\mathrm{Perm}(G')$. Let us check that N is regular if and only if so is $\varphi(N)$. Since they have the same order as $\overline{G}$, it is enough to check that if N is transitive, so is $\varphi(N)$. Let $\tau\,|_F, \tau'\,|_F \in \overline{G}$. Since N is transitive, there is $\eta \in N$ such that $\eta(\tau) = \tau'$. Then, $\varphi(\eta)(\tau\,|_F) = \eta(\tau)\,|_F = \tau'\,|_F$.

We claim that the following diagram is commutative:

$$\begin{array}{ccc} G' & \xrightarrow{\cdot|_M} & \overline{G} \\ {\scriptstyle \lambda^{G'}}\downarrow & & \downarrow{\scriptstyle \lambda^{\overline{G}}} \\ \mathrm{Perm}(G') & \xrightarrow{\varphi} & \mathrm{Perm}(\overline{G}) \end{array} \tag{2.5}$$

Indeed, if $\tau, \tau' \in G'$,

$$\lambda^{\overline{G}}(\tau\,|_M)(\tau'\,|_M) = \tau\tau'\,|_M = \cdot\,|_M \circ \lambda^{G'}(\tau)(\tau') = \varphi(\lambda^{G'}(\tau))(\tau'\,|_M).$$

Then $\lambda^{\overline{G}}(\tau\,|_M) = \varphi(\lambda^{G'}(\tau))$, whence $\lambda^{\overline{G}} \circ\,|_M = \varphi \circ \lambda^G$, as we wanted. It follows that N is normalized by the image of $\lambda^{G'}$ if and only if $\varphi(N)$ is normalized by the image of $\lambda^{\overline{G}}$. □

We have shown in the proof that there is a group isomorphism $\mathrm{Perm}(G') \cong \mathrm{Perm}(\overline{G})$ such that a subgroup $N \leq \mathrm{Perm}(G')$ gives E/L a Hopf-Galois structure if and only if its image in $\mathrm{Perm}(\overline{G})$ gives M/K a Hopf-Galois structure. Thus, we can modify suitably the map ι to obtain a map $\mathrm{Perm}(J) \times \mathrm{Perm}(\overline{G}) \longrightarrow \mathrm{Perm}(G)$. By abuse of notation, we also call this map ι.

Corollary 2.5.29. *If N_1 is a regular subgroup of $\mathrm{Perm}(J)$ normalized by $\lambda_c(J)$ and N_2 is a regular subgroup of $\mathrm{Perm}(\overline{G})$ normalized by $\lambda^{\overline{G}}(\overline{G})$, then $\iota(N_1 \times N_2)$ is a regular subgroup of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$. Accordingly, from a Hopf-Galois structure on L/K and a Hopf-Galois structure on M/K, we obtain an induced Hopf-Galois structure on E/K.*

The advantage of this approach is that the description of the underlying Hopf algebra and the action of an induced Hopf-Galois structure arise naturally from those of the inducing Hopf-Galois structures.

Proposition 2.5.30. *Let E/K be a Galois extension with group $G = J \rtimes G'$, and call $L = E^{G'}$, $M = L^J$. Let H be an induced Hopf-Galois structure on L/K from inducing Hopf-Galois structures H_1 on L/K and H_2 on M/K.*

1. *There is an isomorphism of K-Hopf algebras $H \longrightarrow H_1 \otimes_K H_2$ between the underlying Hopf algebras of H and $H_1 \otimes_K H_2$.*

2. *Given $w \in H_1$, $\eta \in H_2$, $x \in L$ and $y \in M$,*

$$(w\eta) \cdot (xy) = (w \cdot x)(\eta \cdot y).$$

Proof. 1. Recall from Example 2.2.6 that the functor Φ transforms tensor products of K-algebras into cartesian products of sets, so Ψ does the other way around. Let $N_i = \Phi(H_i)$, $i \in \{1,2\}$. By definition of induced Hopf-Galois structure $\Phi(H) = N_1 \times N_2 = \Phi(H_1) \times \Phi(H_2)$. Applying Ψ, we get an isomorphism of K-Hopf algebras $H \cong H_1 \otimes_K H_2$.

2. Let us write $N_1 = \{\eta_i\}_{i=1}^n$ and $N_2 = \{\mu_j\}_{j=1}^m$. Then,

$$w \in H_1 = E[N_1]^G \Longrightarrow w = \sum_{i=1}^{r} c_i \eta_i, c_i \in E,$$

$$\eta \in H_2 = E[N_2]^G \Longrightarrow \eta = \sum_{j=1}^{u} d_j \mu_j, d_j \in E.$$

Hence,

$$\begin{aligned}
(w\eta) \cdot (xy) &= \left(\sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j \iota(\eta_i, \mu_j) \right) \cdot (xy) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j \iota(\eta_i, \mu_j)^{-1}(\mathrm{Id}_G)(xy) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j \iota(\eta_i^{-1}, \mu_j^{-1})(\mathrm{Id}_G)(xy) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j \eta_i^{-1}(\mathrm{Id}_J)(x) \mu_j^{-1}(\mathrm{Id}_{G'})(y) \\
&= \left(\sum_{i=1}^{n} c_i \eta_i^{-1}(\mathrm{Id}_J)(x) \right) \left(\sum_{j=1}^{m} d_j \mu_j^{-1}(\mathrm{Id}_{G'})(y) \right) \\
&= (w \cdot x)(\eta \cdot y)
\end{aligned}$$

□

Remark 2.5.31. If L/K is any H-Galois extension, then we can define a K-linear map $\rho_H \colon H \longrightarrow \mathrm{End}_K(L)$ by $\rho_H(h)(x) = h \cdot x$. In the case that E/K is a Galois extension with group $G = J \rtimes G'$ and $H = H_1 \otimes_K H_2$ is an induced Hopf-Galois structure on E/K, Proposition 2.5.30 2 means that $\rho_H = \rho_{H_1} \otimes_K \rho_{H_2}$.

6 Exercises

6.1 Exercises on Sections 1-3

1. Give a direct proof that the fixed "set" $\mathrm{Fix}(A, H')$ under a sub-Hopf algebra $H' \subset H$ defined in Section 1 is a subalgebra of the H-Galois extension A.

2. (a) The symmetric group S_n of order $n!$ acts naturally on the set $\{1, \ldots, n\}$. What is the stabilizer of an element in that set? (You may take the element n, for example.)

(b) The linear group $GL(n,K)$ acts naturally on the n-dimensional column space K^n. Describe the stabilizer of a non-zero column vector. (You may take for instance the first standard basis vector.)

(c) The special orthogonal group $SO(3,\mathbb{R})$ acts on $\mathbb{R}^3$. Can you describe the stabilizer of $e_1 = (1,0,0)^T$?

3. Assume that the group Γ acts on a set S, and that $s, t \in S$ are in the same orbit. Describe the relation between the stabilizer Γ_s of s and the stabilizer Γ_t of t. Is there any relation in general if s and t are in different orbits?

4. Suppose that $\Gamma = \Gamma_K$ acts on a finite set S such that the open subgroup U of finite index acts trivially. Show there is a normal subgroup $U' < U$ which is still open of finite index in Γ. (Hint: consider conjugates of U.)

5. In Section 2.5 we defined a certain Hopf algebra H^* and specific elements $c, s \in H^*$. Show that the two elements 1 and $h := (1,-1,1,-1) = c^2 - s^2$ are the only group-like elements in H^*. What is the automorphism of L induced by h?

6. Find a base field K and a degree five polynomial f over it, such that the Galois group of f (i.e. of the splitting field of f) is A_5. Note that this splitting field is the normal closure of the extension L/K obtained by adjoining a root of f. (Any means are allowed: the literature, the Internet, your own ideas.)

7. In th context of Lemma 2.3.4, find two more equivalent conditions, now involving right translations ρ_w, with w in the group X.

8. Let C be a cyclic group of order p. Show that every group between $C = C \rtimes 1$ and $\mathrm{Hol}(C) = C \rtimes \mathrm{Aut}(C)$ has the form $C \rtimes \Delta$, where Δ is cyclic and $r = |\Delta|$ divides $p-1$. Do all these r actually occur?

6.2 Exercises on Sections 4-5

1. Let L/K be a (G,G')-separable Hopf-Galois extension and let N_1, N_2 be regular and G-stable subgroups of $\mathrm{Perm}(G/G')$. Consider the action $*$ of G on N_1 and N_2 defined as conjugation by $\lambda(G)$. Show that $N_1 \cong N_2$ as G-groups (that is, there is a G-equivariant group isomorphism between them) if and only if $\widetilde{L}[N_1]^G \cong \widetilde{L}[N_2]^G$ as K-Hopf algebras.

2. Two Hopf-Galois structures $(H,\cdot)$, $(H',\cdot')$ on the same field extension L/K are said to be isomorphic if there is an isomorphism of K-Hopf algebras $f\colon H \longrightarrow H'$ such that $h \cdot a = f(h) \cdot' a$. In practice, isomorphic Hopf-Galois structures on L/K are considered as the same Hopf-Galois structure (for instance, in the Greither-Pareigis theorem).

(a) Let L/K be a Galois extension with group G. Prove that the classical Galois structure on L/K and the Hopf-Galois structure corresponding to $\rho(G)$ under the Greither-Pareigis correspondence are isomorphic.

(b) Give an example of separable Hopf-Galois extension L/K and different (non-isomorphic) Hopf-Galois structures H and H' on L/K such that $N \cong N'$, where N (resp. N') is the permutation subgroup corresponding to H (resp. H').

3. Let L/K be a Galois extension with group G and let N be a regular and G-stable subgroup of $\mathrm{Perm}(G)$. Show that for each $\varphi \in \mathrm{Aut}(G)$, $(\varphi \circ N \circ \varphi^{-1})^{\mathrm{opp}} = \varphi \circ N^{\mathrm{opp}} \circ \varphi^{-1}$.

4. Let L/K be a Galois extension with group G. For each regular and G-stable subgroup N of $\mathrm{Perm}(G)$ and each $g \in G$, call $N_g := \rho(g)N\rho(g^{-1})$. Two Hopf-Galois structures on L/K corresponding to permutation subgroups N, N' are said to be ρ-conjugate if $N' = N_g$ for some $g \in G$. Fix such a subgroup N of $\mathrm{Perm}(G)$ and $g \in G$.

   (a) Prove that N_g is indeed a regular and G-stable subgroup of $\mathrm{Perm}(G)$.

   (b) Show that the map $\phi\colon L[N]^G \longrightarrow L[N_g]^G$ defined by $\phi(\sum_{\eta\in N} c_\eta \eta) = \sum_{\eta\in N} c_\eta \rho(g)\eta\rho(g^{-1})$ is an isomorphism of K-Hopf algebras.

   (c) Prove that $N_g^{\mathrm{opp}} = (N^{\mathrm{opp}})_g$.

5. Prove that every separable field extension of degree at most 4 is almost classically Galois.

6. Let L/K be a (G, G')-separable almost classically Galois extension and let J be a normal complement for G' in G. Write $*$ for the action of G on $M[J]$ defined as the Galois action on M and the conjugation by G on J. Show that if J is abelian, then there is an isomorphism of Hopf-Galois structures between

$$M[J]^{G'} = \{h \in M[J] \mid \tau * h = h \text{ for all } \tau \in G'\}$$

   together with its classical action on L and the Hopf-Galois structure on L/K corresponding to $\lambda(J)$.

7. Consider the extension L/K at Example 2.5.7. Determine explicitly the Hopf-Galois structure associated to the permutation subgroup $N = \langle(\overline{1_G}, \overline{\sigma}, \overline{\sigma^2}, \overline{\sigma^3})\rangle$.

8. Let E/K be a Galois extension with group of the form $J \times G'$ and call $L = E^{G'}$. Prove that the classical Galois structure on E/K is the induced Hopf-Galois structure from the classical Galois structures on E/L and L/K.